

Topological Strings on Elliptic Fibrations

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Abstract

We study topological string theory on elliptically fibered Calabi-Yau manifolds using mirror symmetry. We compute higher genus topological string amplitudes and express these in terms of polynomials of functions constructed from the special geometry of the deformation spaces. The polynomials are fixed by the holomorphic anomaly equations supplemented by the expected behavior at special loci in moduli space. We further expand the amplitudes in the base moduli of the elliptic fibration and find that the fiber moduli dependence is captured by a finer polynomial structure in terms of the modular forms of the modular group of the elliptic curve. We further find a recursive equation which captures this finer structure and which can be related to the anomaly equations for correlation functions.

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1 Introduction

Mirror symmetry and topological string theory are a rich source of insights in both mathematics and physics. The A- and B-model topological string theories probe Kähler and complex structure deformation families of two mirror Calabi-Yau (CY) threefolds Z and Z^* and are identified by mirror symmetry. The B-model is more accessible to computations since its deformations are the complex structure deformations of Z^* which are captured by the variation of Hodge

structure. Mirror symmetry is established by providing the mirror maps which are a distinguished set of local coordinates in a given patch of the deformation space. These provide the map to the A-model, since they are naturally associated with deformations of an underlying superconformal field theory and its chiral ring [1].

At special loci in the moduli space, the A-model data provides enumerative information of the CY Z . This is contained in the Gromov-Witten invariants which can be resummed to give integer multiplicities of BPS states in a five-dimensional theory obtained from an M-theory compactification on Z [2, 3]. Moreover, the special geometry governing the deformation spaces allows one to compute the prepotential $F_0(t)$ which governs the exact effective action of the four dimensional theories obtained from compactifying type IIA(IIB) string theory on $Z(Z^*)$ respectively.

The prepotential is the genus zero free energy of topological string theory, which is defined perturbatively in a coupling constant governing the higher genus expansion. The partition function of topological string theory indicating its dependence on local coordinates in the deformation space has the form:

$$\mathcal{Z}(t, \bar{t}) = \exp \left(\sum_g \lambda^{2g-2} \mathcal{F}^{(g)}(t, \bar{t}) \right). \quad (1.1)$$

In refs.[4, 5], Bershadsky, Cecotti, Ooguri and Vafa (BCOV) developed the theory and properties of the higher genus topological string free energies putting forward recursive equations, the holomorphic anomaly equations along with a method to solve these in terms of Feynman diagrams. For the full partition function these equations take the form of a heat equation [5, 6] and can be interpreted [6] as describing the background independence of the partition function when the latter is interpreted as a wave function associated to the geometric quantization of $H^3(Z^*)$.

The higher genus free energies of the topological string can be furthermore interpreted as giving certain amplitudes of the physical string theory.¹ The full topological string partition function conjecturally also encodes the information of $4d$ BPS states [8]. It is thus natural to expect the topological string free energies to be characterized by automorphic forms of the target space duality group. The modularity of the topological string amplitudes was used in [5] to fix the solutions of the anomaly equation. The modularity of the amplitudes is most manifest whenever the modular group is $SL(2, \mathbb{Z})$ or a subgroup thereof. The higher genus generating functions of the Gromov-Witten invariants for the elliptic curve were expressed as polynomials [9, 10] where the polynomial generators were the elements of the ring of almost holomorphic modular forms E_2, E_4 and E_6 [11]. Polynomials of these generators also appear whenever $SL(2, \mathbb{Z})$ is a subgroup of the modular group, as for example in refs. [12, 13, 14, 15]. The relation of topological strings and almost holomorphic modular forms was further explored in [16] (see also [17] and [18]).

¹See ref.[7] for a review.

Using the special geometry of the deformation space a polynomial structure of the higher genus amplitudes in a finite number of generators was proven for the quintic and related one parameter deformation families [19] and generalized to arbitrary target CY manifolds [20]. The polynomial structure supplemented by appropriate boundary conditions enhances the computability of higher genus amplitudes. Moreover the polynomial generators are expected to bridge the gap towards constructing the appropriate modular forms for a given target space duality group which is reflected by the special geometry of the CY manifold.

In this work we use the polynomial construction to study higher genus amplitudes on elliptically fibered CY. The higher genus amplitudes are expressed in terms of a finite number of generators which are constructed from the special geometry of the moduli space of the CY. Expanding the amplitudes of the elliptic fibration in terms of the base moduli allows us to further express the parts of the amplitudes depending on the fiber moduli in terms of the modular forms of $SL(2, \mathbb{Z})$. Together with this refinement of the polynomial structure we find a refined recursion which is the analog of an equation discovered in the context of BPS state counting of a non-critical string [21, 22, 12] and which was conjectured to hold for higher genus topological strings [13, 14].

Writing the topological string amplitudes for the elliptic fibration which we consider in this work as an expansion:

$$F^{(g)}(t_E, t_B) = \sum_n f_n^{(g)}(t_E) q_B^n,$$

where t_E, t_B denote the special coordinates corresponding to the Kähler parameters of the fiber and base of the elliptic fibration respectively, $q_E = e^{2\pi i t_E}, q_B = e^{2\pi i t_B}$, we find that $f_n^{(g)}$ can be written as

$$f_n^{(g)} = P_n^{(g)}(E_2, E_4, E_6) \frac{q_E^{3n/2}}{\eta^{36n}},$$

where $P_n^{(g)}$ denotes a quasi-modular form constructed out of the Eisenstein series E_2, E_4, E_6 of weight $2g - 2 + 18n$, we furthermore find the following recursion:

$$\frac{\partial f_n^{(g)}}{\partial E_2} = -\frac{1}{24} \sum_{h=0}^g \sum_{s=1}^{n-1} s(n-s) f_s^{(h)} f_{n-s}^{(g-h)} + \frac{n(3-n)}{24} f_n^{(g-1)}. \quad (1.2)$$

The outline of this work is as follows. In Section 2 we review some elements of mirror symmetry that allow us to set the stage for our discussion. We present and further develop techniques to identify the flat coordinates on the deformation spaces. In particular, we exhibit a systematic procedure to determine these coordinates at an arbitrary point in the boundary of the moduli space. We proceed in Section 3 with reviewing the holomorphic anomaly equations and how these can be used together with a polynomial construction to solve for higher genus topological string amplitudes. In Section 4 we present the results of the application of the techniques and methods described earlier to an example of an elliptically fibered CY. The dependence on the moduli of the elliptic fiber can be further organized in terms of polynomials

of E_2, E_4 and E_6 order by order in an expansion in the base moduli. We find a recursion (1.2) which captures this structure and relate it to the anomaly equation for the correlation functions of the full geometry. We show that such recursions hold for several examples of elliptic fibrations. We proceed with our conclusions in section 5.

2 Mirror symmetry

In this section we review some aspects of mirror symmetry which we will be using in the following.² To be able to fix the higher genus amplitudes we need a global understanding of mirror symmetry and how it matches expansion loci in the moduli spaces of the mirror manifolds Z and Z^* . We will also review and further develop some methods and techniques on the B-model side along refs.[29, 30, 31, 32, 33, 34, 35, 36] to identify the special set of coordinates which allows an identification with the physical parameters and hence with the A-model side.

2.1 Mirror geometries

The mirror pair of CY 3-folds (Z, Z^*) is given as hypersurfaces in toric ambient spaces (W, W^*) . The mirror symmetry construction of ref.[24] associates the pair (Z, Z^*) to a pair of integral reflexive polyhedra (Δ, Δ^*) .

The A-model geometry

The polyhedron Δ^* is characterized by k relevant integral points ν_i lying in a hyperplane of distance one from the origin in \mathbb{Z}^5 , ν_0 will denote the origin following the conventions of refs. [24, 25]. The k integral points $\nu_i(\Delta^*)$ of the polyhedron Δ^* correspond to homogeneous coordinates u_i on the toric ambient space W and satisfy $n = h^{1,1}(Z)$ linear relations:

$$\sum_{i=0}^{k-1} l_i^a \nu_i = 0, \quad a = 1, \dots, n. \quad (2.1)$$

The integral entries of the vectors l^a for fixed a define the weights l_i^a of the coordinates x_i under the \mathbb{C}^* actions

$$u_i \rightarrow (\lambda_a)^{l_i^a} u_i, \quad \lambda_a \in \mathbb{C}^*.$$

The l_i^a can also be understood as the $U(1)_a$ charges of the fields of the gauged linear sigma model (GLSM) construction associated with the toric variety [37]. The toric variety W is defined as $W \simeq (\mathbb{C}^k - \Xi)/(\mathbb{C}^*)^n$, where Ξ corresponds to an exceptional subset of degenerate orbits. To construct compact hypersurfaces, W is taken to be the total space of the anti-canonical

²See refs.[23, 24, 25] for foundational material as well as the review book [26] for general background on mirror symmetry. Some of the exposition in this section will follow refs.[27, 28]

bundle over a compact toric variety. The compact manifold $Z \subset W$ is defined by introducing a superpotential $\mathcal{W}_Z = u_0 p(u_i)$ in the GLSM, where x_0 is the coordinate on the fiber and $p(u_i)$ a polynomial in the $u_{i>0}$ of degrees $-l_0^a$. At large Kähler volumes, the critical locus is at $u_0 = p(u_i) = 0$ [37].

An example of an elliptic fibration is the compact geometry given by a section of the anti-canonical bundle over the resolved weighted projective space $\mathbb{P}(1, 1, 1, 6, 9)$. Mirror symmetry for this model has been studied in various places following refs.[25, 38]. The charge vectors for this geometry are given by:

$$\begin{aligned} (l^1) &= \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ -6 & 3 & 2 & 1 & 0 & 0 & 0 \end{pmatrix}, \\ (l^2) &= \begin{pmatrix} 0 & 0 & 0 & -3 & 1 & 1 & 1 \end{pmatrix}. \end{aligned} \quad (2.2)$$

The B-model geometry

The B-model geometry $Z^* \subset W^*$ is determined by the mirror symmetry construction of refs.[39, 24] as the vanishing locus of the equation

$$p(Z^*) = \sum_{i=0}^{k-1} a_i y_i = \sum_{\nu_i \in \Delta} a_i X^{\nu_i}, \quad (2.3)$$

where a_i parameterize the complex structure of Z^* , y_i are homogeneous coordinates [39] on W^* and $X_m, m = 1, \dots, 4$ are inhomogeneous coordinates on an open torus $(\mathbb{C}^*)^4 \subset W^*$ and $X^{\nu_i} := \prod_m X_m^{\nu_{i,m}}$ [40]. The relations (2.1) impose the following relations on the homogeneous coordinates

$$\prod_{i=0}^{k-1} y_i^{l_i^a} = 1, \quad a = 1, \dots, n = h^{2,1}(Z^*) = h^{1,1}(Z). \quad (2.4)$$

The important quantity in the B-model is the holomorphic $(3, 0)$ form which is given by:

$$\Omega(a_i) = \text{Res}_{p=0} \frac{1}{p(Z^*)} \prod_{i=1}^4 \frac{dX_i}{X_i}. \quad (2.5)$$

Its periods

$$\pi_\alpha(a_i) = \int_{\gamma^\alpha} \Omega(a_i), \quad \alpha = 0, \dots, 2h^{2,1} + 1$$

are annihilated by an extended system of GKZ [41] differential operators

$$\mathcal{L}(l) = \prod_{l_i > 0} \left(\frac{\partial}{\partial a_i} \right)^{l_i} - \prod_{l_i < 0} \left(\frac{\partial}{\partial a_i} \right)^{-l_i} \quad (2.6)$$

$$\mathcal{Z}_k = \sum_{i=0}^{k-1} \nu_{i,j} \theta_i, \quad j = 1, \dots, 4. \quad \mathcal{Z}_0 = \sum_{i=0}^{k-1} \theta_i + 1, \quad \theta_i = a_i \frac{\partial}{\partial a_i}, \quad (2.7)$$

where l can be a positive integral linear combination of the charge vectors l^a . The equation $\mathcal{L}(l) \pi_0(a_i) = 0$ follows from the definition (2.5). The equations $\mathcal{Z}_k \pi_\alpha(a_i) = 0$ express the invariance of the period integral under the torus action and imply that the period integrals only depend on special combinations of the parameters a_i

$$\pi_\alpha(a_i) \sim \pi_\alpha(z_a), \quad z_a = (-)^{l_0^a} \prod_i a_i^{l_i^a}, \quad (2.8)$$

the $z_a, a = 1, \dots, n$ define local coordinates on the moduli space \mathcal{M} of complex structures of Z^* .

In our example, there is an additional symmetry on \mathcal{M} . Its origin is the fact that the polytope Δ^* has further integral points on facets [25, 38]. They correspond to nonlinear coordinate transformations of the ambient toric variety W . These coordinate transformations can be compensated by transforming the parameters a_i . This yields the symmetry on \mathcal{M}

$$I : (z_1, z_2) \mapsto \left(\frac{1}{432} - z_1, -\frac{z_1^3 z_2}{(\frac{1}{432} - z_1)^3} \right). \quad (2.9)$$

The charge vectors defining the A-model geometry in Equ.(2.2) give the following Picard-Fuchs (PF) operators annihilating $\tilde{\pi}_\alpha(z_i) = a_0 \pi_\alpha(a_i)$:

$$\mathcal{L}_1 = \theta_1(\theta_1 - 3\theta_2) - 12z_1(6\theta_1 + 1)(6\theta_1 + 5), \quad (2.10)$$

$$\mathcal{L}_2 = \theta_2^3 + z_2 \prod_{i=0}^2 (3\theta_2 - \theta_1 + i), \quad \theta_a := z_a \frac{\partial}{\partial z_a}. \quad (2.11)$$

The discriminants of these operators are

$$\begin{aligned} \Delta_1 &= (1 - 432 z_1)^3 - (432 z_1)^3 27 z_2, \\ \Delta_2 &= 1 + 27 z_2, \end{aligned} \quad (2.12)$$

Furthermore, we label the function

$$\Delta_3 = 1 - 432 z_1. \quad (2.13)$$

Note, that $I(\Delta_1) = (432 z_1)^3 \Delta_2$ and $I(\Delta_2) = \frac{\Delta_1}{\Delta_3^2}$, hence the vanishing loci of Δ_1 and Δ_2 are exchanged under the symmetry I .

A further important ingredient of mirror symmetry are the Yukawa couplings which are identified with the genus zero correlators of three chiral fields of the underlying topological field theory, in the B-model these are defined by:³

$$C_{ijk}(x) := \int_{Z^*} \Omega \wedge \partial_i \partial_j \partial_k \Omega, \quad \partial_i := \frac{\partial}{\partial x^i}. \quad (2.14)$$

³We use $x^i, i = 1, \dots, h^{2,1}$ to denote arbitrary coordinates on the moduli space of complex structures and denote a dependence on these collectively by x . We make the distinction to the coordinates defined in Equ.(2.8) which will be identified with the coordinates centered around the large complex structure limiting point in the moduli space.

For the example above these can be computed using the PF operators [25]:

$$\begin{aligned}
C_{111}(z) &= \frac{9}{z_1^3 \Delta_1}, \\
C_{112}(z) &= \frac{3 \Delta_3}{z_1^2 z_2 \Delta_1}, \\
C_{122}(z) &= \frac{\Delta_3^2}{z_1 z_2^2 \Delta_1}, \\
C_{222}(z) &= \frac{9 (\Delta_3^3 + (432 z_1)^3)}{z_2^2 \Delta_1 \Delta_2}.
\end{aligned} \tag{2.15}$$

2.2 Variation of Hodge structure

The Picard-Fuchs equations capture the variation of Hodge structure which describes the geometric realization on the B-model side of the deformation of the $\mathcal{N} = (2, 2)$ superconformal field theory and its chiral ring [29], see also ref [32] for a review. Choosing one member of the deformation family of CY threefolds Z^* characterized by a point in the moduli space \mathcal{M} of complex structures there is a unique holomorphic $(3, 0)$ form $\Omega(x)$ depending on local coordinates in the deformation space.

A variation of complex structure induces a change of the type of the reference $(3, 0)$ form $\Omega(x)$. This change is captured by the variation of Hodge structure. $H^3(Z^*)$ is the fiber of a complex vector bundle over \mathcal{M} equipped with a flat connection ∇ , the Gauss-Manin connection. The fibers of this vector bundle are constant up to monodromy of ∇ . The Hodge decomposition

$$H^3 = \bigoplus_{p=0}^3 H^{3-p,p},$$

varies over \mathcal{M} as the type splitting depends on the complex structure. A way to capture this variation holomorphically is through the Hodge filtration F^p

$$H^3 = F^0 \supset F^1 \supset F^2 \supset F^3 \supset F^4 = 0, \quad F^p = \bigoplus_{q \geq p} H^{q, 3-q} \subset H^3, \tag{2.16}$$

which define holomorphic subbundles $\mathcal{F}^p \rightarrow \mathcal{M}$ whose fibers are F^p . The Gauss-Manin connection on these subbundles has the property $\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes T^* \mathcal{M}$ known as Griffiths transversality. This property allows us to identify derivatives of $\Omega(x) \in F^3$ with elements in the lower filtration spaces. The whole filtration can be spanned by taking multiderivatives of the holomorphic $(3, 0)$ form. Fourth order derivatives can then again be expressed by the elements of the basis, which is reflected by the fact that periods of $\Omega(x)$ are annihilated the Picard-Fuchs system of differential equations of fourth order. The dimensions of the spaces $(F^3, F^2/F^3, F^1/F^2, F^0/F^1)$ are $(1, h^{2,1}, h^{2,1}, 1)$. Elements in these spaces can be obtained by taking derivatives of $\Omega(x)$ w.r.t the moduli. For the example we are discussing a section of the filtration is given by the

following vector $w(x)$ which has $2h^{2,1} + 2$ components:

$$w(x) = (\Omega(x) \quad \theta_1\Omega(x), \theta_2\Omega(x) \quad \theta_1^2\Omega(x), \theta_1\theta_2\Omega(x), \quad \theta_1^2\theta_2\Omega(x))^t. \quad (2.17)$$

where $\theta_i = x^i \frac{\partial}{\partial x^i}$. Using $w(x)$ we can define the period matrix

$$\Pi(x)_\beta^\alpha = \int_{\gamma^\alpha} w(x), \quad \gamma^\alpha \in H_3(Z^*), \quad \alpha, \beta = 0, \dots, 2h^{2,1} + 1, \quad (2.18)$$

the first row of which corresponds to the periods of $\Omega(x)$. The periods are annihilated by the PF operators. We can identify solutions of the PF operators with the periods of $\Omega(x)$. In our example, near the point of maximal unipotent monodromy $z = (z_1, z_2)$, the solutions are given in the Appendix A.

Polarization

The variation of Hodge structure of a family of Calabi–Yau threefolds in addition comes with a polarization, i.e. a nondegenerate integral bilinear form Q which is antisymmetric. This form is defined by $Q(\varphi, \psi) = \int_{Z^*} \varphi \wedge \psi$ for $\varphi, \psi \in H^3$. The polarization satisfies

$$Q(F^p, F^{4-p}) = 0, \quad Q(C\varphi, \bar{\varphi}) > 0 \text{ for } \varphi \neq 0,$$

where C acts by multiplication of i^{p-q} on $H^{p,q}$. Hence, Q is a symplectic form.

Since the space of periods can be identified with the space of solutions to the Picard–Fuchs equations, the symplectic form on $H^3(Z^*)$ should be expressible in terms of a bilinear operator acting on the space of solutions. This approach has been developed in ref.[36]. We will review and employ these techniques in the following.

We want to express the symplectic form Q in terms of the basis (2.17). For this purpose, we define an antisymmetric bilinear differential operator on the space of solutions of the Picard–Fuchs equation as

$$D_1 \wedge D_2(f_1, f_2) = \frac{1}{2} (D_1 f_1 D_2 f_2 - D_2 f_1 D_1 f_2), \quad (2.19)$$

where D_1 and D_2 are arbitrary differential operators with respect to x . Then we can write Q as an antisymmetric bidifferential operator

$$Q(x) = \sum_{k,l} Q_{k,l}(x) D_k(\theta) \wedge D_l(\theta), \quad (2.20)$$

where D_k, D_l run over the basis (2.17) of multi-derivatives used to define the vector $w(x)$ spanning the Hodge filtration. The condition that $Q(x)$ is constant over the moduli space, i.e.

$$\theta_i Q(x) = 0, \quad i = 1, \dots, h^{2,1}. \quad (2.21)$$

imposes constraints on the coefficients $Q_{k,l}(x)$. These lead to a system of algebraic and differential equations for the $Q_{k,l}(x)$. At this point we need to express the higher order differential

operators in terms of the basis (2.17) using the relations such as (A.2) and (A.3). Then this system can be solved up to an overall constant.

In our example, near the point of maximal unipotent monodromy $z = (z_1, z_2)$, we find

$$Q(z) = \frac{1}{3} \Delta_2 \Delta_3 (\theta_1 \wedge \theta_2^2 + \theta_2 \wedge \theta_1 \theta_2) - \Delta_2 \theta_2 \wedge \theta_2^2 - \frac{a_9}{3 \Delta_3} \theta_1 \wedge \theta_1 \theta_2 \\ - \frac{\Delta_1}{3 \Delta_3^2} 1 \wedge \theta_1 \theta_2^2 + \frac{a_{10}}{\Delta_3^2} 1 \wedge \theta_1 \theta_2 + \frac{a_4}{3 \Delta_3^2} 1 \wedge \theta_1 + \frac{20 z_1 a_9}{\Delta_3^2} 1 \wedge \theta_2. \quad (2.22)$$

where a_4 , a_9 and a_{10} are given in (A.6). In the basis of periods (A.7) we then obtain

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ -1/2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.23)$$

Moreover, the invariant definition of the B-model prepotential is given in terms of the natural symplectic form Q on $H^3(Z^*, \mathbb{Z})$. Let $\varpi_i(x)$ be a basis for the periods, then

$$\mathcal{F}^{(0)}(x) = \frac{1}{2} \sum_{i>j} Q(\varpi_i(x), \varpi_j(x)). \quad (2.24)$$

2.3 The Gauss-Manin connection and flat coordinates

The Gauss-Manin connection

The Picard-Fuchs operators (2.10) are equivalent to a first order equation for the period matrix. Using linear combinations of the operators and derivatives thereof, the system can be cast in the form

$$(\theta_i - A_i(x)) \Pi(x)_\beta^\alpha = 0, \quad i = 1, \dots, h^{2,1}, \quad (2.25)$$

which defines the Gauss-Manin connection ∇ . For our example, the matrices $A_i(x)$ near the point of maximal unipotent monodromy are given in the appendix.

There are limiting points in the moduli space of complex structure \mathcal{M} at which the Hodge structure degenerates [42, 26]. These points are of particular interest in the expansion of the topological string amplitudes. In order to describe these limiting points, we assume that there exists a smooth compactification $\overline{\mathcal{M}}$ of \mathcal{M} such the boundary consists of a finite set I of normal crossing divisors $\overline{\mathcal{M}} \setminus \mathcal{M} = \bigcup_{i \in I} D_i$. Along these divisors, the Gauss-Manin connection can acquire regular singularities. This means that, at a point $p \in \bigcap_{i=1}^{h^{2,1}} D_i$, the connection matrix

has at worst a simple pole along D_i . Note that since we defined A_i in (2.25) with θ_i instead of ∂_i this means that matrix $A_i(z)$ is holomorphic at p .

At a regular singularity described by a divisor $D_i = \{y_i = 0\}$ ⁴ we therefore define:

$$\text{Res}_{D_i}(\nabla) = A_i(y)|_{y_i=0}. \quad (2.26)$$

This residue matrix gives the following useful information. The eigenvalues of the monodromy T are $\exp(2\pi i\lambda)$ as λ ranges over the eigenvalues of $\text{Res}(\nabla)$. Furthermore, T is unipotent if and only if $\text{Res}(\nabla)$ has integer eigenvalues. Finally, if no two distinct eigenvalues of $\text{Res}(\nabla)$ differ by an integer, then T is conjugate to $S = \exp(-2\pi i \text{Res}(\nabla))$. These properties allow us to extract the relevant information about the monodromy of ∇ around these boundary divisors. We will see later that this allows us to easily obtain the solutions to the Picard–Fuchs equations at the various boundary points.

The monodromies T_i for all the divisors D_i in the boundary form a group, the monodromy group Γ of the Gauss–Manin connection. This group is a subgroup of $\text{Aut}(H^3(Z^*, \mathbb{Z}))$ preserving the symplectic form Q . Hence, Γ is a subgroup of $\text{Sp}(2h^{2,1} + 2, \mathbb{Z})$. The topological string amplitudes $\mathcal{F}^{(g)}$ are expected to be automorphic with respect to this group.

The point p in the boundary which has been studied usually so far, is the point of maximal unipotent monodromy, also known as the large complex structure limit. From the connection matrices $A_i(x)$ of our example we can immediately get information on the monodromy matrices around the divisors $D_{(1,0)} = \{z_1 = 0\}$ and $D_{(0,1)} = \{z_2 = 0\}$. (For the notation on the divisors see Section 4.2.) We simply consider the matrices $\text{Res}_{\{z_i=0\}} = A_i(z)|_{z_i=0}$ and bring them into Jordan normal form. This yields

$$\text{Res}_{D_{(1,0)}}(\nabla) \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{Res}_{D_{(0,1)}}(\nabla) \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.27)$$

From this we read off that the corresponding monodromy matrices $T_{D_{1,0}}$ and $T_{D_{0,1}}$ satisfy

$$\left(T_{D_{(1,0)}} - 1\right)^4 = 0, \quad \left(T_{D_{(0,1)}} - 1\right)^3 = 0. \quad (2.28)$$

It can be checked that these monodromy matrices satisfy the conditions for a point of maximal unipotent monodromy [38, 26].

⁴We will denote local coordinates near an intersection point of boundary divisors by y , still reserving z for the point of maximal unipotent monodromy.

Flat coordinates

We proceed by discussing a special set of coordinates on the moduli space of complex structure which permit an identification with the physical deformations of the underlying theory. These coordinates are defined within special geometry which was developed studying moduli spaces of $\mathcal{N} = 2$ theories, we follow refs. [1, 43, 23, 44, 29, 30, 31, 32]. Choosing a symplectic basis of 3-cycles $A^I, B_J \in H_3(Z^*)$ and a dual basis α_I, β^J of $H^3(Z^*)$ such that

$$\begin{aligned} A^I \cap B_J &= \delta_J^I = -B_J \cap A^I, \quad A^I \cap A^J = B_I \cap B_J = 0, \\ \int_{A^I} \alpha_J &= \delta_J^I, \quad \int_{B_J} \beta^I = \delta_J^I, \quad I, J = 0, \dots, h^{2,1}(Z^*), \end{aligned} \quad (2.29)$$

the $(3, 0)$ form $\Omega(x)$ can be expanded in the basis α_I, β^J :

$$\Omega(x) = X^I(x)\alpha_I - \mathcal{F}_J(x)\beta^J. \quad (2.30)$$

The periods $X^I(x)$ can be identified with projective coordinates on \mathcal{M} and \mathcal{F}_J with derivatives of a function $\mathcal{F}(X^I)$, $\mathcal{F}_J = \frac{\partial \mathcal{F}(X^I)}{\partial X^J}$. In a patch where $X^0(x) \neq 0$ a set of special coordinates can be defined

$$t^a = \frac{X^a}{X^0}, \quad a = 1, \dots, h^{2,1}(Z^*).$$

The normalized holomorphic $(3, 0)$ $v_0 = (X^0)^{-1}\Omega(t)$ has the expansion:

$$v_0 = \alpha_0 + t^a \alpha_a - \beta^b F_b(t) - (2F_0(t) - t^c F_c(t))\beta^0, \quad (2.31)$$

where

$$F_0(t) = (X^0)^{-2}\mathcal{F} \quad \text{and} \quad F_a(t) := \partial_a F_0(t) = \frac{\partial F_0(t)}{\partial t^a}.$$

$F_0(t)$ is the prepotential. We define further

$$v_a = \alpha_a - \beta^b F_{ab}(t) - (F_a(t) - t^b F_{ab}(t))\beta^0, \quad (2.32)$$

$$v_D^a = -\beta^a - t^a \beta^0, \quad (2.33)$$

$$v^0 = \beta^0. \quad (2.34)$$

The Yukawa coupling in special coordinates is given by

$$C_{abc} := \partial_a \partial_b \partial_c F_0(t) = \int_{Z^*} v_0 \wedge \partial_a \partial_b \partial_c v_0. \quad (2.35)$$

We further define the vector with $2h^{2,1} + 2$ components:

$$v = (v_0, \quad v_a, \quad v_D^a, \quad v^0)^t, \quad (2.36)$$

We have then by construction:

$$\partial_a \begin{pmatrix} v_0 \\ v_b \\ v_D^b \\ v^0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \delta_a^c & 0 & 0 \\ 0 & 0 & C_{abc} & 0 \\ 0 & 0 & 0 & \delta_a^b \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{:=C_a} \begin{pmatrix} v_0 \\ v_c \\ v_D^c \\ v^0 \end{pmatrix}, \quad (2.37)$$

which defines the $(2h^{2,1} + 2) \times (2h^{2,1} + 2)$ matrices C_a , in terms of which we can write the equation in the form:

$$(\partial_a - C_a) v = 0. \quad (2.38)$$

The entries of v correspond to elements in the different filtration spaces discussed earlier. As in (2.25), Equ.(2.38) defines the Gauss-Manin connection, now in special coordinates. The upper triangular structure of the connection matrix reflects the effect of the charge increment of the elements in the chiral ring upon insertion of a marginal operator of unit charge. Since the underlying superconformal field theory is isomorphic for the A- and the B-models, this set of coordinates describing the variation of Hodge structure is the good one for describing mirror symmetry and provide thus the mirror maps. The following discussion builds on refs.[33, 34, 35].

In order to find the mirror maps starting from a set of arbitrary local coordinates on the moduli space of complex structure we study the relation between the vectors w of Equ.(2.17) and v spanning the Hodge filtration, these are related by the following change of basis:

$$w(x(t)) = M(x(t))v(t). \quad (2.39)$$

By the fact that this change of basis is filtration-preserving, the matrix $M(x)$ must be lower block-triangular. For concreteness we expose the discussion in the following for $h^{2,1} = 2$:

$$M(x) = \begin{pmatrix} m_{11} & 0 & 0 & 0 & 0 & 0 \\ m_{21} & m_{22} & m_{23} & 0 & 0 & 0 \\ m_{31} & m_{32} & m_{33} & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & 0 \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & 0 \\ m_{61} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} \end{pmatrix} \quad (2.40)$$

Imposing that the change of connection matrices yields the desired result requires the vanishing of the following matrix:

$$N_a(t) = C_a(t) - \sum_i J_{ia} M(x)^{-1} (A_i(x)M(x) - \theta_i M(x)) \quad (2.41)$$

has to vanish. Here $J = (J_{ia})$ is the Jacobian for the logarithmic derivative

$$J_{ia} = \frac{1}{x^i} \frac{\partial x^i}{\partial t^a}. \quad (2.42)$$

The matrices N_a have the general block form

$$N_a(x) = \begin{pmatrix} n_{a,11} & n_{a,12} & n_{a,13} & 0 & 0 & 0 \\ n_{a,21} & n_{a,22} & n_{a,23} & n_{a,24} & n_{a,25} & 0 \\ n_{a,31} & n_{a,32} & n_{a,33} & n_{a,34} & n_{a,35} & 0 \\ n_{a,41} & n_{a,42} & n_{a,43} & n_{a,44} & n_{a,45} & n_{a,46} \\ n_{a,51} & n_{a,52} & n_{a,53} & n_{a,54} & n_{a,55} & n_{a,56} \\ n_{a,61} & n_{a,62} & n_{a,63} & n_{a,64} & n_{a,65} & n_{a,66} \end{pmatrix} \quad (2.43)$$

We set $m_{11}(x) = X^0(x)$ since it will become obvious that this quantity should be identified with one of the periods. The vanishing of the first column of the N_a allows us to express the m_{k1} in terms of $X^0(x)$ and its derivatives. Moreover, it follows that m_{11} is a solution to the Picard–Fuchs equations

$$\mathcal{L}_r X^0(x) = 0, \quad (2.44)$$

Similarly, the vanishing of the second and third column of the N_a expresses the m_{k2} and m_{k3} in terms of m_{12} and m_{13} and their derivatives, respectively. In addition, they satisfy differential equations of the form

$$\mathcal{D}_r(t_a X^0) = \mathcal{L}_r(t_a X^0) - t_a \mathcal{L}_r X^0 = 0. \quad (2.45)$$

Together with (2.44) we conclude that the products $t_1 X^0$ and $t_2 X^0$ must be solutions to the Picard–Fuchs equations as well. In other words, the flat coordinates must be ratios of two periods. The differential equations (2.45) form a system of nonlinear partial differential equation which determine the flat coordinates in terms of x . In general, they are hard to solve, but one can transform this system into a system of linear partial differential equations of higher order along the lines of [45].

Next, we consider the blocks $\begin{pmatrix} n_{a,24} & n_{a,25} \\ n_{a,34} & n_{a,35} \end{pmatrix}$. They can be solved for the functions $C_{abc}(t)$. This yields expressions in terms of t_a , their derivatives, and the functions $m_{22}, m_{23}, m_{32}, m_{33}, m_{44}, m_{45}, m_{54}, m_{55}$. Taking into account the previous results, we need to express the latter four functions in terms of X^0 .

The two conditions $n_{a,46}$ can be used to express m_{44} and m_{45} in terms of t_a , their derivatives, and m_{66} . Similarly, $n_{a,56} = 0$ yields similar expression for m_{54} and m_{55} . If we apply this to our example and again choose the point of maximal unipotent monodromy with local coordinates z , then we obtain the following relations

$$\begin{aligned} m_{44}(z) &= \frac{3\theta_2 t_2 - \Delta_3 \theta_1 t_2}{\Delta_3 \det J} m_{66}(z), \\ m_{45}(z) &= -\frac{3\theta_2 t_1 - \Delta_3 \theta_1 t_1}{\Delta_3 \det J} m_{66}(z), \\ m_{54}(z) &= -\frac{(9 - 11664 z_1 + 5038848 z_1^2) \theta_1 t_2 - \Delta_3^2 \Delta_2 \theta_2 t_2}{\Delta_3^2 \Delta_2 \det J} m_{66}(z), \\ m_{55}(z) &= \frac{(9 - 11664 z_1 + 5038848 z_1^2) \theta_1 t_1 - \Delta_3^2 \Delta_2 \theta_2 t_1}{\Delta_3^2 \Delta_2 \det J} m_{66}(z). \end{aligned} \quad (2.46)$$

The vanishing of $n_{1,44}$ and $n_{1,45}$ allows to express m_{64} and m_{65} in terms of m_{42}, \dots, m_{45} , t_a , their derivatives and the C_{abc} . Upon using the previous results, they can be expressed in terms of X^0 , t_i , their derivatives, and $m_{66}(z)$.

To determine the latter, we use the vanishing of the $n_{a,66}$.

$$\frac{432 z_1 (\Delta_1 + 30233088 z_1^2 z_2)}{\Delta_1 \Delta_3} m_{6,6}(z) - \theta_1 m_{6,6}(z) - (\theta_1 t_1) m_{64}(z) - (\theta_1 t_2) m_{65}(z) = 0. \quad (2.47)$$

Substituting all the previous results leads to the following differential equation

$$\Delta_1 \Delta_3 (m_{66}(z) \theta_1 X^0(z) + X^0(z) \theta_1 m_{66}(z)) - 432 z_1 (\Delta_1 + 30233088 z_1^2 z_2) m_{66}(z) X^0(z) = 0. \quad (2.48)$$

All the dependence on the t_i has cancelled. We observe that the prefactor of $m_{66}(z) X^0(z)$ can be written as

$$\frac{\Delta_1^2}{\Delta_3} \theta_1 \left(\frac{\Delta_3^2}{\Delta_1} \right) = 432 z_1 (\Delta_1 + 30233088 z_1^2 z_2). \quad (2.49)$$

Hence, the differential equation simplifies to

$$\theta_1 \left(\frac{\Delta_3^2}{\Delta_1 m_{66}(z) X^0(z)} \right) = 0. \quad (2.50)$$

Its solution is

$$m_{66}(z) = f(z_2) \frac{\Delta_3^2}{\Delta_1 X^0(z)}, \quad (2.51)$$

where $f(z_2)$ is an undetermined function that only depends on z_2 . To fix this function we look at the vanishing of the $n_{2,66}$. After all substitutions this yields the differential equation

$$\theta_2 (\Delta_1 m_{66}(z) X^0(z)) = \theta_2 (f(z_2) \Delta_3^2) = 0. \quad (2.52)$$

Since Δ_3 does not depend on z_2 , we conclude that $f(z_2)$ must be a constant, which we set to 1.

We can now recursively express all the functions m_{ij} through the function $X^0(z)$ which must be a solution of the Picard–Fuchs equations. In particular, this yields the well known expression for the Yukawa couplings in flat coordinates

$$C_{abc}(t) = \sum_{i,j,k} \frac{1}{(X^0(z(t)))^2} \frac{\partial z_i}{\partial t^a} \frac{\partial z_j}{\partial t^b} \frac{\partial z_k}{\partial t^c} C_{ijk}(z(t)). \quad (2.53)$$

There are still a few conditions remaining, namely $n_{a,64} = 0$ and $n_{a,65} = 0$. These turn out to be very difficult to analyze. One can check that these conditions are implied by

$$Q(X^0, t_1 X^0) = 0, \quad Q(X^0, t_2 X^0) = 0, \quad Q(t_1 X^0, t_2 X^0) = 0. \quad (2.54)$$

where Q was determined in (2.22). In particular, not every ratio of solutions to the Picard–Fuchs equations yields a flat coordinate. In general, we expect a weaker condition involving the left-hand sides of (2.54) to be equivalent to the vanishing of N_a .

Solutions of the Picard–Fuchs equations

As we have just seen, in order to determine the flat coordinates we need solutions of the Picard–Fuchs equation which satisfy (2.54). It is well-known how to solve these equations at the point of maximal unipotent monodromy by observing that they form extended GKZ hypergeometric systems, see e.g. [25, 46]. However, we will need the flat coordinates at other special loci in the moduli space. For this purpose we need a systematic procedure to solve the system of Picard–Fuchs equations at an arbitrary point in the boundary $\overline{\mathcal{M}} \setminus \mathcal{M}$ of the moduli space where it is in general no longer of extended GKZ hypergeometric type.

However, if the moduli space \mathcal{M} is one-dimensional we have the following well-known result, see e.g. [47, 48]. Let

$$R = \text{Res}_{y=0} \nabla = A(y)|_{y=0}$$

be the residue matrix of the connection ∇ at a regular singular point given by $y = 0$. R is a constant matrix. Then there exists a fundamental system of solutions to (2.25) of the form

$$u(y) = y^R S(y)$$

with $S(y)$ a single-valued and holomorphic matrix. Since any two fundamental systems are related by an invertible constant matrix, this form is independent of the choice of basis, and we can take for R its Jordan normal form. This simplifies the computations enormously.

In the present case where the moduli space \mathcal{M} is higher-dimensional we can prove the following result: Let $p = \bigcap_{i=1}^n D_i$ be a point at the intersection of $h^{2,1}$ boundary divisors, where each of the divisors D_i is given by an equation $y_i = 0$. Let

$$R_i = \text{Res}_{D_i} \nabla = A_i(y)|_{y_i=0}, \quad \forall i.$$

The matrices R are in general not constant anymore. Then a fundamental system of solutions takes the form

$$u(y) = \prod_{i=1}^n y_i^{R_i} S(y).$$

This follows by induction from the result in dimension 1 together with the fact that $[R_i, R_j] = 0$, a consequence of the flatness of ∇ . Moreover, if J_i is the Jordan normal form of R_i , then there exist constant matrices P_i such that

$$u(y) = \prod_{i=1}^n P_i y_i^{J_i} S(y).$$

This form considerably simplifies the explicit computation. In practice, the P_i are often permutation matrices.

Elliptic fibrations

Here, we discuss a few aspects of elliptic fibrations. Let Z be an elliptically fibered Calabi–Yau threefold $\pi : Z \rightarrow B$ where the fiber $\pi^{-1}(p) \cong E$ is a smooth elliptic curve, $p \in B \setminus \Delta$, where the discriminant Δ is a divisor in B . We consider the variation of Hodge structure for the family of mirror Calabi–Yau threefolds $f : \mathcal{Z}^* \rightarrow \mathcal{M}$ where \mathcal{M} is the complex structure moduli space. We recall that the Gauss–Manin connection for this family has monodromy group $\Gamma \in \text{Aut}(H^3(\mathcal{Z}^*, \mathbb{Z}))$. Since Z is an elliptic fibration, there is a distinguished subgroup of Γ isomorphic to a subgroup $\Gamma_{\text{ell}} \subset \text{SL}_2(\mathbb{Z})$ and the variation of Hodge structure contains a variation of sub–Hodge structures coming from the elliptic fiber.

In our example the monodromy group Γ is generated by two matrices A and T [38]. Consider the element $T_\infty = (TA)^{-1} \in \Gamma$. Then A^3 and T_∞^3 generate an $\text{SL}_2(\mathbb{Z})$ subgroup as follows:

$$A^3 t_1 = -\frac{1}{t_1 + 1}, \quad T_\infty^3 t_1 = t_1 + 1 \quad (2.55)$$

Hence, we expect t_1 to be a modular parameter of an elliptic curve. In fact, in the limit $z_2 \rightarrow 0$ and the Picard–Fuchs system reduces to the Picard–Fuchs equation of the elliptic curve mirror to the elliptic fiber.

3 Higher genus recursion

In this section we review the ingredients of the polynomial construction [19, 20], following [20] as well as the boundary conditions needed to supplement the construction to fix remaining ambiguities. To implement the boundary conditions it is necessary to be able to provide the good physical coordinates in every patch in moduli space, this can be done by exploiting the flat structure of the variation of Hodge structure on the B-model side.

3.1 Special geometry and the holomorphic anomaly

The deformation space \mathcal{M} of topological string theory, parameterized by coordinates x^i , $i = 1, \dots, \dim(\mathcal{M})$, carries the structure of a special Kähler manifold.⁵ The ingredients of this structure are the Hodge line bundle \mathcal{L} over \mathcal{M} and the cubic couplings which are a holomorphic section of $\mathcal{L}^2 \otimes \text{Sym}^3 T^* \mathcal{M}$. The metric on \mathcal{L} is denoted by e^{-K} with respect to some local trivialization and provides a Kähler potential for the special Kähler metric on \mathcal{M} , $G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$. Special geometry further gives the following expression for the curvature of \mathcal{M}

$$R_{i\bar{i}j}^{\bar{l}} = [\bar{\partial}_{\bar{i}}, D_i]^l_j = \bar{\partial}_{\bar{i}} \Gamma_{ij}^l = \delta_i^l G_{j\bar{i}} + \delta_j^l G_{i\bar{i}} - C_{ijk} C_{\bar{i}}^{kl}. \quad (3.1)$$

The topological string amplitude or partition function $\mathcal{F}^{(g)}$ at genus g is a section of the line bundle \mathcal{L}^{2-2g} over \mathcal{M} . The correlation function at genus g with n insertions $\mathcal{F}_{i_1 \dots i_n}^{(g)}$ is only

⁵See ref.[5] for background material.

non-vanishing for $(2g - 2 + n) > 0$. They are related by taking covariant derivatives as this represents insertions of chiral operators in the bulk, e.g. $D_i \mathcal{F}_{i_1 \dots i_n}^{(g)} = \mathcal{F}_{ii_1 \dots i_n}^{(g)}$.

D_i denotes the covariant derivative on the bundle $\mathcal{L}^m \otimes \text{Sym}^n T^* \mathcal{M}$ where m and n follow from the context.⁶ $T^* \mathcal{M}$ is the cotangent bundle of \mathcal{M} with the standard connection coefficients $\Gamma_{jk}^i = G^{i\bar{i}} \partial_j G_{k\bar{i}}$. The connection on the bundle \mathcal{L} is given by the first derivatives of the Kähler potential $K_i = \partial_i K$.

In [5] it is shown that the genus g amplitudes are recursively related to lower genus amplitudes by the holomorphic anomaly equations:

$$\begin{aligned} \bar{\partial}_i \mathcal{F}_{i_1 \dots i_n}^{(g)} &= \frac{1}{2} \bar{C}_i^{jk} \left(\sum_{r=0}^g \sum_{s=0}^n \frac{1}{s!(n-s)!} \sum_{\sigma \in S_n} D_j \mathcal{F}_{i_{\sigma(1)} \dots i_{\sigma(s)}}^{(r)} D_k \mathcal{F}_{i_{\sigma(s+1)} \dots i_{\sigma(n)}}^{(g-r)} + D_j D_k \mathcal{F}_{i_1 \dots i_n}^{(g-1)} \right), \\ &\quad - (2g - 2 + n - 1) \sum_{s=1}^n G_{ii_s} \mathcal{F}_{i_1 \dots i_{s-1} i_{s+1} \dots i_n}^g, \end{aligned} \quad (3.2)$$

where

$$\bar{C}_k^{ij} = \bar{C}_{i\bar{j}\bar{k}} G^{i\bar{i}} G^{j\bar{j}} e^{2K}, \quad \bar{C}_{i\bar{j}\bar{k}} = \overline{C_{ijk}}. \quad (3.3)$$

and where the sum $\sigma \in S_n$ is over permutations of the insertions and the formula is valid for $(g = 0, n \geq 4)$, $(g = 1, n \geq 2)$ and all higher genera and number of insertions. For $n = 0$ it reduces to the holomorphic anomaly for the free energies \mathcal{F}^g :

$$\bar{\partial}_i \mathcal{F}^{(g)} = \frac{1}{2} \bar{C}_i^{jk} \left(\sum_{r=1}^{g-1} D_j \mathcal{F}^{(r)} D_k \mathcal{F}^{(g-r)} + D_j D_k \mathcal{F}^{(g-1)} \right). \quad (3.4)$$

These equations, supplemented by [4]

$$\bar{\partial}_i \mathcal{F}_j^{(1)} = \frac{1}{2} C_{jkl} C_i^{kl} + (1 - \frac{\chi}{24}) G_{j\bar{i}}, \quad (3.5)$$

and special geometry, determine all correlation functions up to holomorphic ambiguities. In Eq. (3.5), χ is the Euler character of the manifold. A solution of the recursion equations is given in terms of Feynman rules [5].

The propagators S , S^i , S^{ij} for these Feynman rules are related to the three point couplings C_{ijk} as

$$\partial_i S^{ij} = \bar{C}_i^{ij}, \quad \partial_i S^j = G_{i\bar{i}} S^{ij}, \quad \partial_i S = G_{i\bar{i}} S^i. \quad (3.6)$$

By definition, the propagators S , S^i and S^{ij} are sections of the bundles $\mathcal{L}^{-2} \otimes \text{Sym}^m T$ with $m = 0, 1, 2$. The vertices of the Feynman rules are given by the correlation functions $\mathcal{F}_{i_1 \dots i_n}^{(g)}$. The anomaly equation Eq. (3.4), as well as the definitions in Eq. (3.6), leave the freedom of adding holomorphic functions under the $\bar{\partial}$ derivatives as integration constants. This freedom is referred to as holomorphic ambiguities.

⁶The notation D_i is also being used for the boundary divisors $D_i \in \overline{\mathcal{M}} \setminus \mathcal{M}$. It is clear from the context which meaning applies.

3.2 Polynomial structure of higher genus amplitudes

In ref.[20] it was proven that the correlation functions $\mathcal{F}_{i_1 \dots i_n}^{(g)}$ are polynomials of degree $3g - 3 + n$ in the generators K_i, S^{ij}, S^i, S where a grading 1, 1, 2, 3 was assigned to these generators respectively. It was furthermore shown that by making a change of generators [20]

$$\begin{aligned}\tilde{S}^{ij} &= S^{ij}, \\ \tilde{S}^i &= S^i - S^{ij} K_j, \\ \tilde{S} &= S - S^i K_i + \frac{1}{2} S^{ij} K_i K_j, \\ \tilde{K}_i &= K_i,\end{aligned}\tag{3.7}$$

the $\mathcal{F}^{(g)}$ do not depend on \tilde{K}_i , i.e. $\partial \mathcal{F}^{(g)} / \partial \tilde{K}_i = 0$. We will henceforth drop the tilde from the modified generators.

The proof relies on expressing the first non-vanishing correlation functions in terms of these generators. At genus zero these are the holomorphic three-point couplings $\mathcal{F}_{ijk}^{(0)} = C_{ijk}$. The holomorphic anomaly equation Eq. (3.4) can be integrated using Eq. (3.6) to

$$\mathcal{F}_i^{(1)} = \frac{1}{2} C_{ijk} S^{jk} + (1 - \frac{\chi}{24}) K_i + f_i^{(1)},\tag{3.8}$$

with ambiguity $f_i^{(1)}$. As can be seen from this expression, the non-holomorphicity of the correlation functions only comes from the generators. Furthermore the special geometry relation (3.1) can be integrated:

$$\Gamma_{ij}^l = \delta_i^l K_j + \delta_j^l K_i - C_{ijk} S^{kl} + s_{ij}^l,\tag{3.9}$$

where s_{ij}^l denote holomorphic functions that are not fixed by the special geometry relation, this can be used to derive the following equations which show the closure of the generators carrying the non-holomorphicity under taking derivatives [20].⁷

$$\begin{aligned}\partial_i S^{jk} &= C_{imn} S^{mj} S^{nk} + \delta_i^j S^k + \delta_i^k S^j - s_{im}^j S^{mk} - s_{im}^k S^{mj} + h_i^{jk}, \\ \partial_i S^j &= C_{imn} S^{mj} S^n + 2\delta_i^j S - s_{im}^j S^m - h_{ik} S^{kj} + h_i^j, \\ \partial_i S &= \frac{1}{2} C_{imn} S^m S^n - h_{ij} S^j + h_i, \\ \partial_i K_j &= K_i K_j - C_{ijn} S^{mn} K_m + s_{ij}^m K_m - C_{ijk} S^k + h_{ij},\end{aligned}\tag{3.10}$$

where h_i^{jk}, h_i^j, h_i and h_{ij} denote holomorphic functions. All these functions together with the functions in Equ.(3.9) are not independent. It was shown in ref. [49] (See also [50]) that the freedom of choosing the holomorphic functions in this ring reduces to functions $\mathcal{E}^{ij}, \mathcal{E}^j, \mathcal{E}$ which can be added to the polynomial generators

$$\hat{S}^{ij} = S^{ij} + \mathcal{E}^{ij},$$

⁷These equations are for the tilded generators of Equ. 3.7 and are obtained straightforwardly from the equations in ref.[20]

$$\begin{aligned}\widehat{S}^j &= S^j + \mathcal{E}^j, \\ \widehat{S} &= S + \mathcal{E}.\end{aligned}\tag{3.11}$$

All the holomorphic quantities change accordingly.

The topological string amplitudes now satisfy the holomorphic anomaly equations where the $\bar{\partial}_i$ derivative is replaced by derivatives with respect to the polynomial generators [20].

$$\begin{aligned}\frac{\partial \mathcal{F}_{i_1 \dots i_n}^{(g)}}{\partial S^{ij}} - \frac{1}{2} \left(K_i \frac{\partial \mathcal{F}_{i_1 \dots i_n}^{(g)}}{\partial S^j} + K_j \frac{\partial \mathcal{F}_{i_1 \dots i_n}^{(g)}}{\partial S^i} \right) + \frac{1}{2} K_i K_j \frac{\partial \mathcal{F}_{i_1 \dots i_n}^{(g)}}{\partial S} = \\ \frac{1}{2} \sum_{r=0}^g \sum_{s=0}^n \frac{1}{s!(n-s)!} \sum_{\sigma \in S_n} D_j \mathcal{F}_{i_{\sigma(1)} \dots i_{\sigma(s)}}^{(r)} D_k \mathcal{F}_{i_{\sigma(s+1)} \dots i_{\sigma(n)}}^{(g-r)} + \frac{1}{2} D_j D_k \mathcal{F}_{i_1 \dots i_n}^{(g-1)}\end{aligned}\tag{3.12}$$

$$\sum_i G_{i\bar{i}} \frac{\partial \mathcal{F}_{i_1 \dots i_n}^{(g)}}{\partial K_i} = -(2g - 2 + n - 1) \sum_{s=1}^n G_{i\bar{i}_s} \mathcal{F}_{i_1 \dots i_{s-1} i_s \dots i_n}^g.\tag{3.13}$$

This equation can be simplified by grouping powers of K_i [50]. For example for $n = 0$ this gives the following set of equations:

$$\begin{aligned}\frac{\partial \mathcal{F}^{(g)}}{\partial S^{ij}} &= \frac{1}{2} \left(\partial_i \partial_j + (C_{ijl} S^{lk} - s_{ij}^k) \partial_k + (4 - 2g)(h_{ij} - C_{ijk} S^k) \right) \mathcal{F}^{(g-1)} + \\ &+ \frac{1}{2} \sum_{h=1}^{g-1} \partial_i \mathcal{F}^{(g-h)} \partial_j \mathcal{F}^{(h)},\end{aligned}\tag{3.14}$$

$$\frac{\partial \mathcal{F}^{(g)}}{\partial S^i} = (3g - 2) \partial_i \mathcal{F}^{(g-1)} + \sum_{h=1}^{g-1} (2h - 2) \partial_i \mathcal{F}^{(g-h)} \mathcal{F}^{(h)},\tag{3.15}$$

$$\frac{\partial \mathcal{F}^{(g)}}{\partial S} = (4 - 2g)(3 - 2g) \mathcal{F}^{(g-1)} + \sum_{h=1}^{g-1} (2 - 2g + 2h)(2 - 2h) \mathcal{F}^{(g-h)} \mathcal{F}^{(h)}.\tag{3.16}$$

3.3 Constructing the generators

The construction of the generators of the polynomial construction has been discussed in ref.[49]. The starting point is to pick a local coordinate z_* on the moduli space such that C_{*ij} is an invertible $n \times n$ matrix in order to rewrite Eq.(3.9) as

$$S^{ij} = (C_*^{-1})^{ik} \left(\delta_*^j K_k + \delta_k^j K_* - \Gamma_{*k}^j + s_{*k}^j \right)\tag{3.17}$$

The freedom in Eq.(3.11) can be used to choose some of the s_{ij}^k [49]. The other generators are then constructed using the equations (3.10) [49]:

$$S^i = \frac{1}{2} \left(\partial_i S^{ii} - C_{imn} S^{mi} S^{mi} + 2s_{im}^i S^{mi} - h_i^{ii} \right),\tag{3.18}$$

$$S = \frac{1}{2} (\partial_i S^i - C_{imn} S^m S^{ni} + s_{im}^i S^m + h_{im} S^{mi} - h_i^i) . \quad (3.19)$$

In both equations the index i is fixed, i.e. there is no summation over that index. The freedom in adding holomorphic functions to the generators of Eq.(3.11) can again be used to make some choice for the functions h_i^{ii}, h_i^i , the other ones are fixed by this choice and can be computed from Eq.(3.10).

3.4 Boundary conditions

Genus 1

The holomorphic anomaly equation at genus 1 (3.5) can be integrated to give:

$$\mathcal{F}^{(1)} = \frac{1}{2} \left(3 + h^{2,1} - \frac{\chi}{12} \right) K + \frac{1}{2} \log \det G^{-1} + \sum_i s_i \log z_i + \sum_a r_a \log \Delta_a , \quad (3.20)$$

where $i = 1, \dots, h^{2,1}$ and a runs over the number of discriminant components. The coefficients s_i and r_a are fixed by the leading singular behavior of $\mathcal{F}^{(1)}$ which is given by [5]

$$\mathcal{F}^{(1)} \sim -\frac{1}{24} \sum_i \log z_i \int_Z c_2 J_i , \quad (3.21)$$

for the algebraic coordinates z_i , for a discriminant Δ corresponding to a conifold singularity the leading behavior is given by

$$\mathcal{F}^{(1)} \sim -\frac{1}{12} \log \Delta . \quad (3.22)$$

Higher genus boundary conditions

The holomorphic ambiguity needed to reconstruct the full topological string amplitudes can be fixed by imposing various boundary conditions for $\mathcal{F}^{(g)}$ at the boundary of the moduli space. As in Section 2.3 we assume that the boundary is described by simple normal crossing divisors $\overline{\mathcal{M}} \setminus \mathcal{M} = \bigcup_{i \in I} D_i$ for some finite set I .

We can distinguish the various boundary conditions by looking at the monodromy T_i of the Gauss–Manin connection ∇ around a boundary divisor D_i . By the monodromy theorem [51] we know that T_i must satisfy

$$(T_i^m - 1)^n = 0 \quad (3.23)$$

for $n \leq \dim Z + 1$ and some positive integer m . The current understanding of the boundary conditions for $\mathcal{F}^{(g)}$ seems to suggest that they can be classified according to the value of n .

The large complex structure limit

A point in the boundary is a large complex structure limit or a point of maximal unipotent monodromy if $n = \dim Z^* + 1$ in (3.23) and if $N_i = \log T_i$ satisfies certain conditions described in detail in [26] and [38].

The leading behavior of $\mathcal{F}^{(g)}$ at this point (which is mirror to the large volume limit) was computed in [4, 5, 52, 2, 53, 3]. In particular the contribution from constant maps is

$$\mathcal{F}^{(g)}|_{q_a=0} = (-1)^g \frac{\chi}{2} \frac{|B_{2g} B_{2g-2}|}{2g(2g-2)(2g-2)!}, \quad g > 1, \quad (3.24)$$

where q_a denote the exponentiated mirror map at this point.

Conifold-like loci

A divisor D_i in the boundary is of conifold type if $n = 2$ in (3.23). If $m = 1$ then Z^* acquires a conifold singularity, if $m > 1$ the singularity is not of conifold type but the physics behaves similarly. This singularity is often called a strong coupling singularity [54]. Singularities of both types appear at the vanishing of the discriminant Δ . A well-known example for the case $m > 1$ is the divisor given by the non-principal discriminant in the moduli space of the mirror of $\mathbb{P}(1, 1, 2, 2, 6)$ [12] for which $m = 2$.

The leading singular behavior of the partition function $\mathcal{F}^{(g)}$ at a conifold locus has been determined in [4, 5, 55, 56, 2, 3]

$$\mathcal{F}^{(g)}(t_c) = b \frac{B_{2g}}{2g(2g-2)t_c^{2g-2}} + O(t_c^0), \quad g > 1 \quad (3.25)$$

Here $t_c \sim \Delta^{\frac{1}{m}}$ is the flat coordinate at the discriminant locus $\Delta = 0$. For a conifold singularity $b = 1$ and $m = 1$. In particular the leading singularity in (3.25) as well as the absence of subleading singular terms follows from the Schwinger loop computation of [2, 3], which computes the effect of the extra massless hypermultiplet in the space-time theory [57]. The singular structure and the “gap” of subleading singular terms have been also observed in the dual matrix model [58] and were first used in [59, 60] to fix the holomorphic ambiguity at higher genus. The space-time derivation of [2, 3] is not restricted to the conifold case and applies also to the case $m > 1$ singularities which give rise to a different spectrum of extra massless vector and hypermultiplets in space-time. The coefficient of the Schwinger loop integral is a weighted trace over the spin of the particles [57, 56] leading to the prediction $b = n_H - n_V$ for the coefficient of the leading singular term. The appearance of the prefactor b in the case $m > 1$ has been discussed in [49] for the case of the local \mathbb{F}_2 (see also [61]).

Orbifold loci

A divisor D_i in the boundary is of orbifold type if $n = 1$ in (3.23). In this case, the monodromy is of finite order. The leading singular behavior of the partition function $\mathcal{F}^{(g)}$ at a such a divisor

is expected to be regular [5]

$$\mathcal{F}^{(g)}(t_o) = O(t_o^0), \quad g > 1. \quad (3.26)$$

where t_o is the flat coordinate at the orbifold locus D_i .

The holomorphic ambiguity

The singular behavior of $\mathcal{F}^{(g)}$ is taken into account by the local ansatz

$$\text{hol.ambiguity} \sim \frac{p(\tilde{z}_i)}{\Delta^{(2g-2)}}, \quad (3.27)$$

for the holomorphic ambiguity near $\Delta = 0$, where $p(\tilde{z}_i)$ is a priori a series in the local coordinates \tilde{z}_i near the singularity. Patching together the local information at all the singularities with the boundary divisors with finite monodromy, it follows however that the numerator $p(z_i)$ is generically a polynomial of low degree in the z_i . Here z_i denote the natural coordinates centered at large complex structure, $z_i = 0 \ \forall i$. The finite number of coefficients in $p(z_i)$ is constrained by (3.25).

4 Higher genus amplitudes for an elliptic fibration

In this section we use the polynomial construction together with the boundary conditions discussed previously to construct the higher genus topological string amplitudes for the example of the elliptic fibration which we discussed.

4.1 Setup of the polynomials

We start by setting up the polynomial construction as discussed in section 3.2. This involves using the freedom in choosing the generators in order to fix the holomorphic functions appearing in the derivative relations (3.10). We fix the choice of the polynomial generators such that these functions are rational expressions in terms of the coordinates in the large complex structure patch of the moduli space. For the holomorphic functions in the following we multiply lower indices by the corresponding coordinates and divide by the coordinates corresponding to upper indices.

$$A_i^j \rightarrow \frac{z_i}{z_j} A_i^j$$

With this convention we can express all the holomorphic functions appearing in the setup of the polynomial construction in terms of polynomials in the local coordinates. We start by fixing the choice of the generators S^{ij} in Eqs.(3.17,3.9):

$$s_{11}^1 = -2, \quad s_{12}^1 = -\frac{1}{3}, \quad s_{22}^1 = 0, \quad (4.1)$$

$$s_{11}^2 = 0, \quad s_{12}^2 = 0, \quad s_{22}^2 = -\frac{4}{3}. \quad (4.2)$$

Then the following quantities are partly chosen by fixing the choice of the generators S^i in Equ.(3.18) and the other quantities are then computed from Eqs.(3.10);

$$h_1^{11} = \frac{1}{9} - 48 z_1 + \frac{5}{6} z_2 - 540 z_1 z_2, \quad (4.3)$$

$$h_1^{12} = -\frac{5}{108} - \frac{5}{4} z_2 + 20 z_1 + 540 z_1 z_2, \quad (4.4)$$

$$h_1^{22} = -60 z_1 (1 - 27 z_2), \quad (4.5)$$

$$h_2^{11} = -60 z_1 z_2, \quad (4.6)$$

$$h_2^{12} = \frac{1}{9} + \frac{5}{12} z_2 - 48 z_1, \quad (4.7)$$

$$h_2^{22} = -\frac{23}{54} + 40 z_1 - \frac{5}{2} z_2 - 540 z_1 z_2. \quad (4.8)$$

We proceed by fixing the choice of the generator S in Equ.(3.18) and compute from Equ.(3.10)

$$h_1^1 = \frac{155}{27} z_1 - \frac{25}{1296} z_2 + 50 z_1 z_2, \quad (4.9)$$

$$h_1^2 = 0, \quad (4.10)$$

$$h_2^1 = -\frac{5}{18} z_2 + 120 z_1 z_2, \quad (4.11)$$

$$h_2^2 = \frac{155}{27} z_1 + \frac{1055}{1296} z_2 + 50 z_1 z_2. \quad (4.12)$$

we further compute:

$$h^1 = \frac{25}{23328}, \quad h^2 = -\frac{50}{3} z_1 z_2. \quad (4.13)$$

and

$$h_{11} = \frac{5}{36}, \quad h_{12} = \frac{5}{108}, \quad h_{22} = 0. \quad (4.14)$$

With these choices the polynomial part of the higher genus amplitudes is entirely fixed by equations (3.14,3.15,3.16). However we need to supplement this polynomial part with the holomorphic ambiguities which are not captured by the holomorphic anomaly recursion and can be fixed by the boundary conditions discussed earlier. In order to implement the boundary conditions we make an ansatz for the ambiguities which will be discussed later. We then expand the polynomial part and the ansatz in the local special coordinates in the different patches of moduli space. In order to do this for the discussed example we first proceed by discussing the moduli space and its various loci.

4.2 Moduli space and its compactification

To obtain a nice and useful description of the moduli space of complex structures, we first need the secondary fan of the variety W . This is obtained from the columns of the Mori generators

(2.2) which are (taking the primitive lattice vectors in \mathbb{Z}^2)

$$b_1 = (1, 0), \quad b_2 = (0, 1), \quad b_3 = (1, -3), \quad b_4 = (-1, 0). \quad (4.15)$$

These vectors define the weighted projective space $\mathbb{P}(1, 1, 3)$ blown up in one point, with toric divisors $D_{(1,0)}$, $D_{(0,1)}$, $D_{(1,-3)}$, $D_{(-1,0)}$, respectively. (The divisor $D_{(1,-3)}$ does not lie on the boundary of the moduli space [38] and will be neglected in the following.) This space is still singular, and we will discuss the resolution of the singularities in the next subsection.

We still have to remove the set where the hypersurface is singular, i. e. the discriminant locus. This is also given in terms of the data of toric geometry as follows: If θ is any face of the polytope Δ^* , we define $f_\theta(x) = \sum_{\nu_i \in \theta \cap \mathbb{Z}^4} a_i \prod_i X^{\nu_i}$. The polynomial is degenerate if for any face $\theta \subset \Delta^*$, the system of polynomial equations

$$f_\theta = X_1 \frac{\partial f}{\partial X_1} = \cdots = X_4 \frac{\partial f}{\partial X_4} = 0 \quad (4.16)$$

has no solution in the toric variety. This yields that the discriminant locus is given by the divisors

$$D_1 = \{\Delta_1 = 0\}, \quad D_2 = \{\Delta_2 = 0\}. \quad (4.17)$$

with Δ_1 and Δ_2 given in (2.12).

In the following we will use the following abbreviations

$$\bar{z}_1 = 432z_1, \quad \bar{z}_2 = -27z_2 \quad (4.18)$$

These divisors intersect each other as follows. From $\Delta_1 = (1 - \bar{z}_1)^3 - \bar{z}_1^2 \bar{z}_2$, we see that there is a tangency of order 3 between $D_{(0,1)}$ and D_1 at the point $(1, 0)$. Writing $\Delta_1 = (1 - 3\bar{z}_1 + 3\bar{z}_1^2) + \bar{z}_1^3 \Delta_2$ we see that there is a triple intersection of D_1 and D_2 intersect transversally in the two points $(\bar{z}_1, \bar{z}_2) = (\bar{z}_\pm, 1)$ with $\bar{z}_\pm = \frac{1}{2} \left(1 \pm i\sqrt{3}\right)$. By changing to the variables to $w_1 = \frac{1}{\bar{z}_1}$ we write $\Delta_1 = -w_1(3 - 3w_1 + w_1^2) + \Delta_2$ and we have a triple intersection of D_1 , D_2 , and $D_{(-1,0)}$ in $(w_1, \bar{z}_2) = (0, 1)$.

Resolution of singularities

We want a compactification of the complex structure moduli space by divisors with normal crossings. To achieve this we must resolve the singularities of $\mathbb{P}(1, 1, 3)$ and resolve all nonnormal crossings of D_1 and D_2 with any of the other divisors. Moreover, we will need a set of local coordinates near of each normal crossing.

The singularities of $\mathbb{P}(1, 1, 3)$ can be taken care of by toric geometry. Resolving them amounts to subdividing the secondary fan and this introduces three further generators $b_5 = (1, -1)$, $b_6 = (1, -2)$ and $b_7 = (0, -1)$, and the corresponding toric divisors $D_{(1,-1)}$, $D_{(1,-2)}$ and $D_{(0,-1)}$. Toric geometry also provides us with the local coordinates near each intersection point of the toric divisors. They are determined by the generators of the cone dual to the maximal cone

spanned by the corresponding generators. E.g. The dual cone to $\langle 0, b_5, b_6 \rangle$ is spanned by the vectors $(2, 1)$ and $(-1, -1)$, hence the corresponding local coordinates are $\left(\bar{z}_1^2 \bar{z}_2, \frac{1}{\bar{z}_1 \bar{z}_2}\right)$. A summary can be found in Table 1.

In order to obtain normal crossings with D_1 and D_2 we first consider the resolution of the singularity of the hypersurface $W = x^3 - y^4 = 0$ in $(0, 0)$. We view the hypersurface $W = 0$ as a divisor D in \mathbb{C}^2 . The resolution can be performed in terms of four blow-ups.

At the first blow-up, we introduce a \mathbb{P}^1 with homogeneous coordinates $(u_0 : v_0)$ such that $u_0 x - v_0 y = 0$. We denote this exceptional divisor by E_0 . In the coordinate patch $u_0 = 1$ we have $x = v_0 y$ and the singularity becomes $W = y^3(v_0^3 - y)$. $W = 0$ now consists of the components $E_0 = \{y = 0\}$ and $D = \{v_0^3 - y = 0\}$ which do not intersect transversely in $(v_0, y) = (0, 0)$. On the other hand, in the coordinate patch $v_0 = 1$, we have $y = u_0 x$ and the singularity becomes $W = x^3(1 - u_0^4 x)$. $W = 0$ consists of the components $E_0 = \{x = 0\}$ and $D = \{1 - u_0^4 x = 0\}$ which do not intersect at all. Hence, we focus on the patch $u_0 = 1$ with local coordinates (v_0, y) and resolve further.

At the second blow-up, we introduce a \mathbb{P}^1 with homogeneous coordinates (u_1, v_1) such that $u_1 v_0 - v_1 y = 0$. We denote this exceptional divisor by E_1 . In the coordinate patch $u_1 = 1$ we have $v_0 = v_1 y$ and the singularity becomes $W = y^4(v_1^3 y^2 - 1)$. $W = 0$ now consists of the components $E_1 = \{y = 0\}$ and $D = \{v_1^2 y^2 - 1 = 0\}$ which do not intersect. On the other hand, in the coordinate patch $v_1 = 1$, we have $y = u_1 v_0$ and the singularity becomes $W = u_1^3 v_0^4(v_0^2 - u_2)$. $W = 0$ consists of the components $E_1 = \{v_0 = 0\}$, $E_0 = \{u_1 = 0\}$ and $D = \{v_0^2 - u_1 = 0\}$ which do not intersect transversely in $(v_0, u_1) = (0, 0)$. Hence, we focus on the patch $v_1 = 1$ with local coordinates (v_0, u_1) and resolve further.

At the third blow-up, we introduce a \mathbb{P}^1 with homogeneous coordinates $(u_2 : v_2)$ such that $u_2 v_0 - v_2 u_1 = 0$. We denote this exceptional divisor by E_2 . In the coordinate patch $u_2 = 1$ we have $v_0 = v_2 u_1$ and the singularity becomes $W = u_1^6 v_2^2(u_1 v_2^2 - 1)$. $W = 0$ consists of the components $E_2 = \{u_1 = 0\}$, $E_1 = \{v_2 = 0\}$ and $D = \{u_1 v_2^2 - 1 = 0\}$ which do not intersect. On the other hand, in the coordinate patch $v_2 = 1$, we have $u_1 = u_2 v_0$ and the singularity becomes $W = u_2^3 v_0^6(v_0 - u_2)$. $W = 0$ consists of the components $E_2 = \{v_0 = 0\}$, $E_0 = \{u_2 = 0\}$ and $D = \{v_0 - u_2 = 0\}$ which do not intersect transversely in $(v_0, u_2) = (0, 0)$. Hence, we focus on the patch $v_2 = 1$ with local coordinates (v_0, u_2) and resolve further.

At the fourth and final blow-up, we introduce a \mathbb{P}^1 with homogeneous coordinates $(u_3 : v_3)$ such that $u_3 v_0 - v_3 u_2 = 0$. We denote this exceptional divisor by E_3 . In the coordinate patch $u_3 = 1$ we have $v_0 = v_3 u_2$ and the singularity becomes $W = u_2^{10} v_3^6(v_3 - 1)$. $W = 0$ consists of the components $E_3 = \{u_2 = 0\}$, $E_2 = \{v_3 = 0\}$, $D = \{v_3 - 1 = 0\}$ which do not intersect. On the other hand, in the coordinate patch $v_3 = 1$, we have $u_4 = u_3 v_0$ and the singularity becomes $W = u_3^3 v_0^{10}(1 - u_3)$. $W = 0$ consists of the components $E_3 = \{v_0 = 0\}$, $E_0 = \{u_3 = 0\}$ and $D = \{1 - u_3 = 0\}$ which do intersect transversely in $(u_3, v_0) = (0, 0)$. Hence, we have completely resolved the singularity.

We see that $E_0 \cap E_3 = \{v_0 = 0, u_3 = 0\}$, $E_3 \cap D = \{v_0 = 0, u_3 = 1\} = \{u_2 = 0, v_3 = 1\}$

and $E_0 \cap D = \emptyset$. Moreover, in the other patch, $E_3 \cap E_2 = \{u_2 = 0, v_3 = 0\}$, $E_2 \cap D = \emptyset$, and $E_0 \cap E_2 = \emptyset$. Since E_1 does not appear anymore, $E_3 \cap E_1 = \emptyset$, its intersections can only be seen in the previous patch with coordinates (u_1, v_2) and are $E_0 \cap E_1 = \emptyset$ and $E_1 \cap E_2 = \{u_1 = 0, v_2 = 0\}$.

Now, we apply this to the divisors in the moduli space of the mirror of $\mathbb{P}(1, 1, 1, 6, 9)$ [18]. After the first blow-up $W = v_0^3 - y$ describes a tangency of order 3 which locally can be identified with the tangency of D_1 and $D_{(0,1)}$. This yields $D = D_1$, $E_0 = D_{(0,1)}$ with local coordinates

$$v_0 = 1 - \bar{z}_1, \quad y = -\bar{z}_1^3 \bar{z}_2.$$

From this we get

$$\begin{aligned} u_1 &= \frac{y}{v_0} = -\frac{\bar{z}_1^3 \bar{z}_2}{1 - \bar{z}_1}, & v_1 &= \frac{v_0}{y} = -\frac{1 - \bar{z}_1}{\bar{z}_1^3 \bar{z}_2}, \\ u_2 &= \frac{u_1}{v_0} = -\frac{\bar{z}_1^3 \bar{z}_2}{(1 - \bar{z}_1)^2}, & v_2 &= \frac{v_0}{u_1} = -\frac{(1 - \bar{z}_1)^2}{\bar{z}_1^3 \bar{z}_2}, \\ u_3 &= \frac{u_2}{v_0} = -\frac{\bar{z}_1^3 \bar{z}_2}{(1 - \bar{z}_1)^3}, & v_3 &= \frac{v_0}{u_2} = -\frac{(1 - \bar{z}_1)^3}{\bar{z}_1^3 \bar{z}_2}. \end{aligned} \quad (4.19)$$

With these identifications we find for the local coordinates near the four intersections of these divisors

$$\begin{aligned} D_1 \cap E_3 &: \left(1 + \frac{\bar{z}_1^3 \bar{z}_2}{(1 - \bar{z}_1)^3}, 1 - \bar{z}_1\right) \\ D_{(0,1)} \cap E_3 &: \left(-\frac{\bar{z}_1^3 \bar{z}_2}{(1 - \bar{z}_1)^3}, 1 - \bar{z}_1\right) \\ E_2 \cap E_3 &: \left(-\frac{\bar{z}_1^3 \bar{z}_2}{(1 - \bar{z}_1)^2}, -\frac{(1 - \bar{z}_1)^3}{\bar{z}_1^3 \bar{z}_2}\right) \\ E_1 \cap E_2 &: \left(-\frac{\bar{z}_1^3 \bar{z}_2}{1 - \bar{z}_1}, -\frac{(1 - \bar{z}_1)^2}{\bar{z}_1^3 \bar{z}_2}\right) \end{aligned} \quad (4.20)$$

Similarly, the triple intersection $W = u_2 v_0 (v_0 - u_2)$ after the third blowup locally can be identified with the triple intersection of D_1 , D_2 , and $D_{(-1,0)}$. For this purpose, we set

$$u_2 = 1 - \bar{z}_2, \quad v_0 = \alpha w_1$$

where $\alpha = w_1^2 - 3w_1 + 3$ which is nonzero at $w_1 = 0$. This yields $D = D_1$, $E_0 = D_2$ and $E_2 = D_{(-1,0)}$. From this we get (recalling $w_1 = \frac{1}{\bar{z}_1}$ neglecting factors of α)

$$u_3 = \frac{u_2}{v_0} = \bar{z}_1(1 - \bar{z}_2), \quad v_3 = \frac{v_0}{u_2} = \frac{1}{\bar{z}_1(1 - \bar{z}_2)}. \quad (4.21)$$

Relabeling the exceptional divisor E_3 by E_0 we find for the local coordinates near the two

intersection points

$$\begin{aligned}
D_1 \cap E_0 &: \left(\frac{1}{\bar{z}_1^2} ((1 - \bar{z}_1)^3 + \bar{z}_1^3 \bar{z}_2), \frac{1}{\bar{z}_1} \right) \\
D_2 \cap E_0 &: \left(\bar{z}_1(1 - \bar{z}_2), \frac{1}{\bar{z}_1} \right) \\
D_{(0,1)} \cap E_0 &: \left(\frac{1}{\bar{z}_1(1 - \bar{z}_2)}, 1 - \bar{z}_2 \right)
\end{aligned} \tag{4.22}$$

This concludes the construction of the compactification of the moduli space with normal crossing divisors. We summarize the local coordinates in Table 1. where $\bar{z}_\pm = \frac{1}{2} \pm i\frac{\sqrt{3}}{6}$. We give a sketch

Crossing	Local coordinates
$D_{(1,0)} \cap D_{(0,1)}$	(\bar{z}_1, \bar{z}_2)
$D_{(1,0)} \cap D_{(1,-1)}$	$(\bar{z}_1 \bar{z}_2, \bar{z}_2^{-1})$
$D_{(1,0)} \cap D_2$	$(\bar{z}_1, 1 - \bar{z}_2)$
$D_{(1,-2)} \cap D_{(1,-1)}$	$((\bar{z}_1 \bar{z}_2)^{-1}, \bar{z}_1^2 \bar{z}_2)$
$D_2 \cap E_0$	$(\bar{z}_1(1 - \bar{z}_2), \frac{1}{\bar{z}_1})$
$D_{(-1,0)} \cap D_{(0,-1)}$	$(\bar{z}_1^{-1}, \bar{z}_2^{-1})$
$D_{(-1,0)} \cap D_{(0,1)}$	$(\bar{z}_1^{-1}, \bar{z}_2)$
$D_{(-1,0)} \cap E_0$	$(\frac{1}{\bar{z}_1(1 - \bar{z}_2)}, 1 - \bar{z}_2)$
$(D_1 \cap D_2)_+$	$\left(1 - \frac{\bar{z}_1}{\bar{z}_+}, \frac{1 - \bar{z}_2}{1 - \frac{\bar{z}_1}{\bar{z}_+}} \right)$
$(D_1 \cap D_2)_-$	$\left(1 - \frac{\bar{z}_1}{\bar{z}_-}, \frac{1 - \bar{z}_2}{1 - \frac{\bar{z}_1}{\bar{z}_-}} \right)$
$D_1 \cap E_0$	$\left(\frac{(1 - \bar{z}_1)^3 + \bar{z}_1^3 \bar{z}_2}{\bar{z}_1^2}, \frac{1}{\bar{z}_1} \right)$
$E_3 \cap E_2$	$\left(-\frac{\bar{z}_1^3 \bar{z}_2}{(1 - \bar{z}_1)^2}, -\frac{(1 - \bar{z}_1)^3}{\bar{z}_1^3 \bar{z}_2} \right)$
$E_3 \cap D_{(0,1)}$	$\left(1 - \bar{z}_1, -\frac{\bar{z}_1^3 \bar{z}_2}{(1 - \bar{z}_1)^3} \right)$
$E_3 \cap D_1$	$\left(1 - \bar{z}_1, 1 + \frac{\bar{z}_1^3 \bar{z}_2}{(1 - \bar{z}_1)^3} \right)$
$E_1 \cap E_2$	$\left(-\frac{\bar{z}_1^3 \bar{z}_2}{1 - \bar{z}_1}, -\frac{(1 - \bar{z}_1)^2}{\bar{z}_1^3 \bar{z}_2} \right)$

Table 1:

for the compactification in Figure 1. The divisor $D_{(1,-3)}$ is drawn with a dashed line since it not in the boundary of the moduli space. Under the action of the symmetry I given in (2.9),

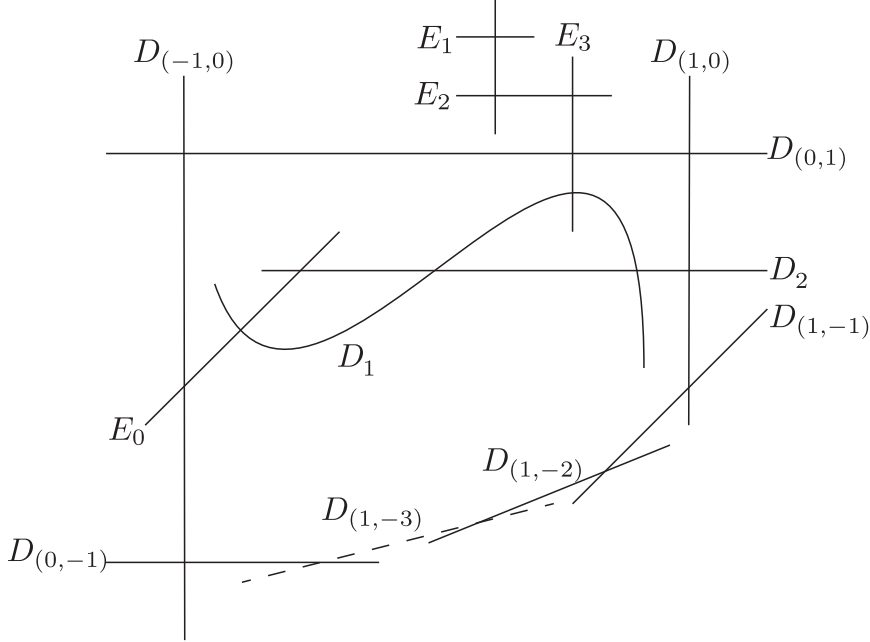


Figure 1: The blown-up moduli space

we have

$$\begin{aligned} I(D_{(1,-2)}) &= E_1, & I(D_{(1,-1)}) &= E_2, & I(D_{(1,0)}) &= E_3, & I(D_1) &= D_2, \\ I(D_{(0,1)}) &= D_{(0,1)}, & I(D_{(0,-1)}) &= D_{(0,-1)}, & I(D_{(-1,0)}) &= D_{(-1,0)}, & I(E_0) &= E_0. \end{aligned} \quad (4.23)$$

For a sketch of the compactification in coordinates in which this symmetry becomes manifest, see [38].

4.3 Periods and flat coordinates at the boundary points

Consider the intersection points p of the boundary divisors listed in Table 1. We again denote the local coordinates near one of these points p by y . For each of the first nine intersections p of (the remaining ones can be obtained by applying the symmetry I) we determine the Gauss–Manin connection. This can be done in two ways, starting from the results at the large complex structure point reviewed in Section 2. Either one performs the change of variables from z to y given in this table in the Picard–Fuchs equation (2.10) and then reads off the connection matrix as discussed in Appendix A, or one transforms the connection matrix using the gauge transformation law for this change of variables. In both cases, one needs to specify a basis of periods near the intersection of interest. We choose it to be the same everywhere and as in (2.17) and express it in terms of differential operators acting on a period as

$$1, \quad \theta_1, \quad \theta_2, \quad \theta_1^2, \quad \theta_1\theta_2, \quad \theta_1^2\theta_2. \quad (4.24)$$

where $\theta_i = y_i \frac{\partial}{\partial y_i}$.

Once we have the connection matrices $A_i(y)$, we can determine their residues. The residues are then used in two ways. First, they allow us to compute the index of the monodromy about the divisors intersecting p . Second, they enter into the solutions of the Picard–Fuchs equations as discussed in Section 2. For the residues we find (the residues for $D_{(1,0)}$ and $D_{(0,1)}$ have been displayed in (2.27))

$$\begin{aligned}
\text{Res}_{D_{(1,-1)}} &\sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \end{pmatrix} & \text{Res}_{D_2} &\sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & (4.25) \\
\text{Res}_{D_{(1,-2)}} &\sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} & \text{Res}_{E_0} &\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{11}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
\text{Res}_{D_{(-1,0)}} &\sim \begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} \end{pmatrix} & \text{Res}_{D_{(0,-1)}} &\sim \begin{pmatrix} \frac{1}{18} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{18} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{18} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{11}{18} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{13}{18} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{17}{18} \end{pmatrix} \\
\text{Res}_{D_1} &\sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \text{Res}_{E_3} &\sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\text{Res}_{E_2} \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \end{pmatrix} \quad \text{Res}_{E_1} \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

We note that the monodromy matrices Res_{D_1} , Res_{D_2} appear at the various intersection points always with an eigenvalue 1 of multiplicity 3, but the multiplicities of the eigenvalues 0 and 2 are different at different points of intersection. This does not matter here, and can easily be remedied by multiplying the basis elements (4.24) with appropriate powers of y_i . We have summarized the behaviour of the various monodromy matrices in Table 2. (This has first been obtained in [38]. The monodromies about $D_{(1,-1)}$ and $D_{(1,-2)}$ can be related to the one about $D_{(1,0)}$ through the local toric geometry [62].) We note here that by [38] the generators of the

$D_{(1,0)}$	$(T-1)^4 = 0$
$D_{(0,1)}$	$(T-1)^3 = 0$
$D_{(1,-1)}$	$(T^3-1)^4 = 0$
$D_{(1,-2)}$	$(T^3-1)^4 = 0$
$D_{(0,-1)}$	$T^{18} - 1 = 0$
$D_{(-1,0)}$	$T^6 - 1 = 0$
D_1	$(T-1)^2 = 0$
D_2	$(T-1)^2 = 0$
E_0	$T^6 - 1 = 0$
E_1	$(T^3-1)^4 = 0$
E_2	$(T^3-1)^4 = 0$
E_3	$(T-1)^4 = 0$

Table 2:

monodromy group Γ are $D_{(0,-1)}$ and D_1 . The generators of the monodromy subgroup Γ_{ell} corresponding to the elliptic fiber are $D_{(1,0)}$ and $D_{(0,-1)}$ ³.

For the solutions of the Picard–Fuchs equations we only give an example, for the other points the results are analogous. The local coordinates at the intersection $D_{(1,0)} \cap D_{(1,-1)}$ read

$$y_1 = -11664 z_1 z_2, \quad y_2 = -\frac{1}{27 z_2} \quad (4.26)$$

The residue matrices at $y_i = 0$ have been displayed in (4.25). The solutions of the Picard–Fuchs

operators take the form

$$\begin{aligned}
\pi_0(y) &= s_0(y) \\
\pi_1(y) &= s_0 \log \left(y_1 y_2^{\frac{2}{3}} \right) + s_1(y) \\
\pi_2(y) &= s_0 \log \left(y_1 y_2^{\frac{2}{3}} \right)^2 + 2 s_1(y) \log \left(y_1 y_2^{\frac{2}{3}} \right) + s_2(y) \\
\pi_3(y) &= s_0 \log \left(y_1 y_2^{\frac{2}{3}} \right)^3 + 3 s_1 \log \left(y_1 y_2^{\frac{2}{3}} \right)^2 + 3 s_2(y) \log \left(y_1 y_2^{\frac{2}{3}} \right) + s_3(y) \\
\pi_4(y) &= y_2^{\frac{1}{3}} s_4(y) \\
\pi_5(y) &= y_2^{\frac{2}{3}} s_5(y)
\end{aligned} \tag{4.27}$$

with

$$\begin{aligned}
s_0(y) &= 1 + \frac{5}{36} y_1 y_2 h^2 + O(h^4) \\
s_1(y) &= \frac{31}{36} y_1 y_2 h^2 + O(h^4) \\
s_2(y) &= \frac{5}{18} y_1 y_2 h^2 + O(h^4) \\
s_3(y) &= -y_2 h + \left(-\frac{9}{40} y_2^2 - \frac{5}{6} y_1 y_2 \right) h^2 + O(h^3) \\
s_4(y) &= 1 + \frac{1}{24} y_2 h + \left(\frac{4}{315} y_2^2 + \frac{5}{72} y_1 y_2 \right) h^2 + O(h^3) \\
s_5(y) &= 1 + \left(-\frac{5}{18} y_1 + \frac{2}{15} y_2 \right) h + O(h^2)
\end{aligned} \tag{4.28}$$

We obtain the symplectic form Q at p in the same way as the connection matrices A_i , by changing the variables in (2.22). Then, inserting the solutions $\pi_i(y)$ yields the intersection form

$$Q = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & -\frac{1}{27} & 0 & 0 & 0 \\ 0 & \frac{1}{27} & 0 & 0 & 0 & 0 \\ -\frac{1}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{27} \\ 0 & 0 & 0 & 0 & -\frac{1}{27} & 0 \end{pmatrix} \tag{4.29}$$

This allows us to choose the flat coordinates as follows:

$$\begin{aligned}
t_1(y) &= \frac{\pi_1(y)}{\pi_0(y)} = \log \left(y_1 y_2^{\frac{2}{3}} \right) + \frac{31}{36} y_1 y_2 h^2 + O(h^4) \\
t_2(y) &= \frac{\pi_4(y)}{\pi_0(y)} = y_2^{\frac{1}{3}} \left(1 + \frac{1}{24} y_2 h + O(h^2) \right)
\end{aligned} \tag{4.30}$$

The partition function for $g = 2, 3$

Having the flat coordinates at all the intersections points at the boundary at hand, we can proceed to apply the boundary conditions discussed in Section 3. In genus 1, we use $c_2 J_1 = 102$ and $c_2 J_2 = 36$ to fix the s_i in (3.20), and more find $r_1 = r_2 = -\frac{1}{6}$.

From Table 2 we see that $D_{(0,-1)}$, $D_{(-1,0)}$, and E_0 are of orbifold type, while D_1 and D_2 are of conifold type.

The condition that $\mathcal{F}^{(g)}$ be regular at a divisor with finite monodromy, i.e. at $D_{(0,-1)}$, and $D_{(-1,0)}$ ensures that the holomorphic function $p^{(g)}(z)$ is a polynomial. The degrees (d_1, d_2) of its monomials are bounded by

$$d_1 \leq 7(g-1), \quad d_2 \leq 6(g-1) - 1, \quad 3d_2 - d_1 \leq 9(g-1). \quad (4.31)$$

In addition, regularity at $D_{(1,-1)}$ fixes the coefficients of the monomials with degrees $3d_2 - d_1 > 3(2g-2)$, while regularity at $D_{(-1,0)}$ fixes those with $d_1 > 6(g-1)$. The divisor E_0 does not yield additional conditions.

Since D_1 and D_2 are of conifold type, we can use the expansion (3.25). In order to do so, we have to take into account that the flat coordinates t_1, t_2 obtained from the above process are only determined up to normalization. Hence we expect relations $t_i = k_i t_{c,i}$, $i = 1, 2$, where $t_{c,i}$ are the flat coordinates in the expansion (3.25). The gap condition from this expansion yields an overdetermined systems of relations among the remaining coefficients of $p^{(g)}(z)$. This system has a unique solution depending only on the parameter k_1 . This normalization factor could in principle be determined by an explicit analytic continuation of the periods $\pi(z)$ at the large complex structure limit to the periods $\pi(y)$ at $D_1 \cap D_2$, though this is highly complicated.

Finally, at the large complex structure limit we can apply the Gopakumar–Vafa expansion [3]:

$$\mathcal{F}(Z, t, \lambda) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{\beta} \sum_{m>0} \sum_{r \geq 0} \frac{1}{m} n_{\beta}^{(g)}(Z) \left(2 \sin\left(\frac{m\lambda}{2}\right)\right)^{2g-2} q^{m\beta}$$

where $c(t)$ and $l(t)$ are a cubic and linear polynomials, respectively, depending on topological invariants of Z . Using the fact that there are no degree 1 curves of genus 2 in the base, $n_{0,1}^{(2)} = 0$ allows us to determine k_1 as well. The Gopakumar–Vafa invariants $n_{\beta}^{(g)}$ are listed in Appendix C. The resulting expressions for the ambiguities $f^{(2)}(z)$ and $f^{(3)}(z)$ can be found in Appendix B. For $g > 3$ the computations turn out to be too involved. Moreover, we expect that the boundary conditions known so far, will not be sufficient to fix the holomorphic ambiguity. We observe that the $\mathcal{F}^{(g)}$ also show a particular behaviour at the other boundary divisors D_i , however, it is not possible to formulate it just from the resulting expression.

4.4 Recursion in terms of modular forms of $\text{SL}(2, \mathbb{Z})$

Having computed the topological string partition function up to genus 3 we proceed in the following with exploring the manifestation of the $\text{SL}(2, \mathbb{Z})$ subgroup of the modular group.

To do so we examine the large complex structure expansion of $\mathcal{F}^{(g)}$ in terms of the special coordinates. We need further to choose a section of the corresponding line bundle \mathcal{L}^{2-2g} . We do so by fixing the gauge $\pi_0(z) = 1$, where π_0 is the analytic solution at large complex structure given in Equ.(A.7). The special, flat coordinates in this patch of moduli space are given by

$$t_E := t_1 = \frac{\pi_1}{\pi_0}, \quad t_B := t_2 = \frac{\pi_2}{\pi_0}, \quad q_E := q_1 = e^{2\pi i t_1}, \quad q_B := q_2 = e^{2\pi i t_2}. \quad (4.32)$$

where the periods π_i are given in the Appendix A. We consider the functions

$$F^{(g)}(t_E, t_B) = \pi_0(z(t))^{2g-2} \mathcal{F}^{(g)}(z(t)), \quad (4.33)$$

and expand these in the exponentiated base modulus q_B :

$$F^{(g)}(t_E, t_B) = \sum_n f_n^{(g)}(t_E) q_B^n = \sum_n \frac{1}{n!} \frac{\partial^n F^{(g)}}{\partial q_B^n} \Big|_{q_B=0} q_B^n, \quad (4.34)$$

we find that $f_n^{(g)}$ can be written as

$$f_n^{(g)} = P_n^{(g)}(E_2, E_4, E_6) \frac{q_E^{3n/2}}{\Delta^{3n/2}},$$

where $P_n^{(g)}$ denotes a quasi-modular form constructed out of the Eisenstein series E_2, E_4, E_6 of weight $2g + 18n - 2$, some examples of these are given in appendix (D.2) we furthermore find that the f_n^g satisfy the following recursion:

$$\frac{\partial f_n^{(g)}}{\partial E_2} = -\frac{1}{24} \sum_{h=0}^g \sum_{s=1}^{n-1} s(n-s) f_s^{(h)} f_{n-s}^{(g-h)} + \frac{n(3-n)}{24} f_n^{(g-1)}. \quad (4.35)$$

This recursion is analogous to a recursion which was conjectured for higher genus in refs. [13, 14]. The geometry considered in these works was that of a $\frac{1}{2}K_3$. The recursion at genus 0 was motivated by a recursion in the BPS state counting of the non-critical string [21, 22, 12] and its relation to the prepotential of the geometry used to construct these [63].⁸

The recursion at genus zero can be verified explicitly either from the construction of the polynomial expressions from integrals of the underlying Seiberg-Witten type curve [21, 22] or from the properties of the Picard-Fuchs equations [13]. The higher genus version of the equation is verified for low genera by the explicit construction of the polynomials. In particular, the explicit knowledge of the holomorphic ambiguities $f^{(2)}$ and $f^{(3)}$ allow us to determine the E_2 independent part of the polynomials $P_n^{(g)}$ which is not determined by (4.35). Moreover, the higher genus version is conjectured to be equivalent to the BCOV anomaly equation [14, 50]. In the following we want to relate qualitatively the equation (4.35) to the anomaly equations for the amplitudes with insertions in its polynomial form (3.12, 3.13).

⁸More recently this geometry has also been studied in [64].

We work with the coordinates centered at large complex structure z_1 and z_2 and consider the free energy with n insertions w.r.t z_2 :

$$F_n^{(g)} := (\pi_0)^{2g-2} \mathcal{F}_{\underbrace{2 \dots 2}_n}^{(g)}$$

since z_2 is not the flat coordinate, the insertions are defined using covariant derivatives on $T^*\mathcal{M}$. We will use however that $z_2 = q_2 + \dots$ and hence to leading order derivatives w.r.t q_2 are captured by the amplitudes with insertions w.r.t z_2 .

We are now interested in the appearance of E_2 in the $q_2 \rightarrow 0$ limit in the polynomial generators of the full problem, we find an occurrence in two of the generators:⁹

$$\left(\frac{S^{22}}{z_2^2} \right) \Big|_{q_2=0} = -\frac{1}{12} E_2 + E_4^{1/2} + \frac{1}{12} \frac{E_6}{E_4}, \quad (4.36)$$

$$K_1 \Big|_{q_2=0} = \frac{E_4^{3/2}}{\Delta} (E_2 E_4 - E_6). \quad (4.37)$$

We hence have

$$\frac{\partial F_n^{(g)}}{\partial E_2} \Big|_{q_2=0} = \left(\frac{\partial F_n^{(g)}}{\partial S^{22}} \frac{\partial S^{22}}{\partial E_2} + \frac{\partial F_n^{(g)}}{\partial K_1} \frac{\partial K_1}{\partial E_2} \right) \Big|_{q_2=0}, \quad (4.38)$$

the two terms on the r.h.s can be computed from Eqs.(3.12,3.13). The second of which vanishes in this case due to the vanishing of the Kähler metric G_{12} on the r.h.s of Equ.(3.13) in the limit $q_2 \rightarrow 0$.

We therefore have from (3.12):

$$\frac{\partial F_n^{(g)}}{\partial S^{22}} = \frac{1}{2} \sum_{h=0}^g \sum_{s=0}^n D_2 F_s^{(h)} D_2 F_{n-s}^{(g-h)} + \frac{1}{2} D_2 D_2 F_n^{(g-1)} \quad (4.39)$$

and furthermore:

$$\frac{\partial F_n^{(g)}}{\partial E_2} \Big|_{q_2=0} = -\frac{z_2^2}{24} \left(\sum_{h=0}^g \sum_{s=0}^n D_2 F_s^{(h)} D_2 F_{n-s}^{(g-h)} + D_2 D_2 F_n^{(g-1)} \right) \Big|_{q_2=0} \quad (4.40)$$

We further compute $z_2 \Gamma_{22}^2 \Big|_{q_2=0} = -1$ and note that

$$z_2 D_2 F_n^{(g)} \Big|_{q_2=0} = \left(\theta_2 F_n^{(g)} - n z_2 \Gamma_{22}^2 F_n^{(g)} \right) \Big|_{q_2=0} = n \left(F_n^{(g)} \right) \Big|_{q_2=0},$$

Relating the $f_n^{(g)} \sim F_n^{(g)} \Big|_{q_2=0}$ it is possible to see the characteristic traits of equation (4.35). Due to the multiplication with z_2^2 the non-vanishing contribution of the first term on the r.h.s of (4.40) is coming from the product of the connections with prefactor $s(n-s)$, from the second term, a contribution of $n(n+1)$ is coming from the contribution of the product of the

⁹Since S^{22} is a section of \mathcal{L}^{-2} we fix a section by multiplying by π_0^2 , we moreover have $\pi_0 \Big|_{q_2=0} = E_4^{1/4}$.

two connections. Further contributions come from derivatives acting on the connections. This completes our qualitative relation of refined recursion (4.35) to the polynomial form of the holomorphic anomaly equation with insertions (3.12). A more thorough matching of the two equations is beyond the scope of this work and will be discussed elsewhere.

4.5 Further examples

The expansion (4.35) also holds for other elliptic fibrations. We present here some more examples. The first is a section of the anti-canonical bundle over the resolved weighted projective space $\mathbb{P}(1, 1, 1, 3, 6)$. The charge vectors for this geometry are given by:

$$\begin{aligned} (l^1) &= \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ -4 & 2 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \\ (l^2) &= \begin{pmatrix} 0 & 0 & 0 & -3 & 1 & 1 & 1 \end{pmatrix}. \end{aligned} \quad (4.41)$$

If we take the derivative with respect to $E_2(2\tau)$ instead of $E_2(\tau)$, then (4.35) holds with the first initial condition given as

$$f_1^{(0)} = \frac{3}{8} F_2 G_2^3 (16 F_2^4 - 51 F_2^2 G_2^2 + 51 G_2^4) \Delta^{-3/2}, \quad (4.42)$$

where F_2 and G_2 are modular forms of weight 2 and generate the ring of modular forms for $\Gamma(2)$. They can be expressed in terms of Jacobi theta functions as

$$\begin{aligned} F_2(\tau) &= \theta_2(\tau)^4 + \theta_3(\tau)^4, \\ G_2(\tau) &= \theta_2(\tau)^4 - \theta_3(\tau)^4. \end{aligned} \quad (4.43)$$

The same is true, if we consider a section of the anti-canonical bundle over the resolved weighted projective space $\mathbb{P}(1, 1, 1, 3, 3)$ whose charge vectors are

$$\begin{aligned} (l^1) &= \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ -3 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \\ (l^2) &= \begin{pmatrix} 0 & 0 & 0 & -3 & 1 & 1 & 1 \end{pmatrix}. \end{aligned} \quad (4.44)$$

Take the derivative with respect to $E_2(3\tau)$ instead of $E_2(\tau)$, then (4.35) holds with initial condition

$$f_1^{(0)} = 9 E_1 (E_1^6 - 87 F_3 E_1^3 + 2349 F_3^2) (E_1^3 - 27 F_3)^3 \Delta^{-3/2}, \quad (4.45)$$

where E_1 and F_3 are modular forms of weight 1 and 3, respectively, and generate the ring of modular forms for $\Gamma_1(3)$. They can be expressed in terms of the Dedekind eta functions as

$$\begin{aligned} E_1(\tau) &= \frac{(\eta(\tau)^{12} + 27\eta(3\tau)^{12})^{\frac{1}{3}}}{\eta(\tau)\eta(3\tau)}, \\ F_3(\tau) &= \frac{\eta(3\tau)^9}{\eta(\tau)^3}. \end{aligned} \quad (4.46)$$

Another elliptic fibration whose associated congruence subgroup is $\Gamma_1(3)$ is the degree $(3, 3)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$. Its charge vectors are

$$\begin{aligned} (l^1) &= \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ -3 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \\ (l^2) &= \begin{pmatrix} -3 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}. \end{aligned} \quad (4.47)$$

and the first initial condition for the recursion is

$$F_1^{(0)} = 27 E_1 (7 E_1^3 + 54 F_3) \Delta^{-1/2}. \quad (4.48)$$

The argument of the previous subsection also applies to elliptic fibrations over Hirzebruch surfaces \mathbb{F}_n . They have more than one base modulus. For example, the elliptic fibration given by the charge vectors

$$\begin{aligned} (l^1) &= \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ -6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ (l^2) &= \begin{pmatrix} 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \end{pmatrix}, \\ (l^3) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \end{aligned} \quad (4.49)$$

has base \mathbb{F}_0 . In this case the recursion (4.35) takes the following form

$$\begin{aligned} \frac{\partial f_{m,n}^{(g)}}{\partial E_2} &= -\frac{1}{24} (2mn - 2m - 2n) f_{m,n}^{(g-1)} \\ &\quad - \frac{1}{24} \sum_{h=0}^g \sum_{s=0}^m \sum_{t=0}^n (s(n-t) + t(m-s)) f_{s,t}^{(g-h)} f_{m-s,n-t}^{(h)} \end{aligned} \quad (4.50)$$

with first initial condition

$$f_{0,1}^{(0)} = -2 \frac{E_4 E_6}{\Delta}. \quad (4.51)$$

and $f_{m,n}^{(g)} = f_{n,m}^{(g)}$. The fact that the $f_{m,n}^{(g)}$ can be expressed in the form $f_{m,n}^{(g)} = P_{m,n}^{(g)}(E_2, E_4, E_6) \Delta^{-m-n}$ where $P_{m,n}^{(g)}(E_2, E_4, E_6)$ is a quasi-modular form of weight $2g - 2 + 12m + 12n$ has already been observed in [15].

Next, we consider an elliptic fibration over the Hirzebruch surface \mathbb{F}_1 which has two phases. In the phase with charge vectors

$$\begin{aligned} (l^1) &= \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ -6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ (l^2) &= \begin{pmatrix} 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \end{pmatrix}, \\ (l^3) &= \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 \end{pmatrix}. \end{aligned} \quad (4.52)$$

the recursion turns out to be

$$\begin{aligned} \frac{\partial f_{m,n}^{(g)}}{\partial E_2} &= -\frac{1}{24} (2mn - 2m - 2n) f_{m,n}^{(g-1)} \\ &\quad + \frac{1}{24} \sum_{h=0}^g \sum_{s=0}^m \sum_{t=0}^n (t(n-t) - s(n-t) - t(m-s)) f_{s,t}^{(g-h)} f_{m-s,n-t}^{(h)} \end{aligned} \quad (4.53)$$

with first initial condition

$$f_{0,1}^{(0)} = \frac{E_4}{\Delta^{1/2}} \quad (4.54)$$

In this case, the quasi-modular form $P_{m,n}^{(g)}(E_2, E_4, E_6)$ has weight $2g - 2 + 12m + 6n$. The modularity of $f_{0,1}^{(0)}$ has been analyzed in detail in [63].

Finally, for the elliptic fibration over \mathbb{F}_2 given by the charge vectors

$$\begin{aligned} & \begin{matrix} & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{matrix} \\ (l^1) = & \begin{pmatrix} -6 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ (l^2) = & \begin{pmatrix} 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \end{pmatrix}, \\ (l^3) = & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 \end{pmatrix}. \end{aligned} \quad (4.55)$$

we find that the recursion turns out to be

$$\begin{aligned} \frac{\partial f_{m,n}^{(g)}}{\partial E_2} = & -\frac{1}{24} (2mn - 2m - 2n) f_{m,n}^{(g-1)} \\ & + \frac{1}{24} \sum_{h=0}^g \sum_{s=0}^m \sum_{t=0}^n (2t(n-t) - s(n-t) - t(m-s)) f_{s,t}^{(g-h)} f_{m-s,n-t}^{(h)} \end{aligned} \quad (4.56)$$

with first initial condition

$$f_{1,0}^{(0)} = -2 \frac{E_4 E_6}{\Delta}, \quad f_{0,1}^{(0)} = 0. \quad (4.57)$$

In this case, the quasi-modular form $P_{m,n}^{(g)}(E_2, E_4, E_6)$ has weight $2g - 2 + 12m$.

5 Conclusions

In this work we studied topological string theory and mirror symmetry on an elliptically fibered CY. We computed higher genus amplitudes for this geometry using their polynomial structure and appropriate boundary conditions. The implementation of the boundary conditions required the use of techniques to single out the preferred coordinates on the deformation space of complex structures on the B-model side of topological strings. To do this we used the Gauss-Manin connection and the special, flat coordinates which could be found in various loci in the moduli space. At the large volume limiting point on the A-side which is mirror to the B-model large complex structure limit, the topological string free energies reduce to the Gromov-Witten generating functions allowing us thus to make predictions for these invariants at genus 2 and 3 in their resummed version giving the GV integer BPS degeneracies.

Having computed the higher genus topological string amplitudes we showed that these carry an additional interesting structure which exhibits the elliptic fibration. Namely the order by order expansion in terms of the moduli of the base of the elliptic fibration can be expressed in terms of the characteristic modular forms of $\text{SL}(2, \mathbb{Z})$ which is a subgroup of the full modular group due to the elliptic fibration. Along with this refined expansion in terms of E_2, E_4 and

E_6 we found a refined anomaly equation which could be related to the holomorphic anomaly equations of BCOV for the correlation functions. This type of anomaly is the analog of an anomaly which was studied in the study of BPS states of exceptional non-critical strings [21, 22, 12] which are captured by the prepotential of the geometry used in their construction [63]. It was furthermore shown in [12] that this anomaly is related to an anomaly found in the study of partition functions of $\mathcal{N} = 4$ topological SYM theory [65]. The anomaly for the that latter theory on \mathbb{P}^2 found in [65] marks the first physical appearance of what became to be know as mock modular forms (See ref.[66] for an introduction). The relation of the non-holomorphicity of mock modular forms and the recursion at genus 0 was further studied in [67, 68, 69, 70]. The recursion found in this work (4.35) is expected to shed more light on the higher rank $\mathcal{N} = 4$ topological SYM theory on \mathbb{P}^2 , since the main example of this paper is an elliptic fibration over \mathbb{P}^2 and the elliptic fibration structure is the analogous setup to ref.[12]. It would be furthermore interesting to give the higher genus amplitudes an interpretation in the SYM theory.

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A Gauss-Manin connection matrices

The vector $w(z)$ with $2h^{2,1} + 2$ components:

$$w(z) = \left(\Omega(z) \quad \theta_1 \Omega(z), \theta_2 \Omega(z) \quad \theta_1^2 \Omega(z), \theta_1 \theta_2 \Omega(z), \quad \theta_1^2 \theta_2 \Omega(z) \right)^t. \quad (\text{A.1})$$

was picked such that its entries span the filtration quotient groups $(F^3, F^2/F^3, F^1/F^2, F^0/F^1)$ of respective orders $(1, h^{2,1}, h^{2,1}, 1)$. Further multiderivatives of $\Omega(z)$ can be expressed in terms of the elements of this vector using the Picard-Fuchs equations, derivatives and linear combinations thereof. We find the following relation for the remaining double derivative:

$$\theta_1^2 = \frac{3(\theta_2 \theta_1 + 144 z_1 \theta_1 + 20 z_1)}{\Delta_3}. \quad (\text{A.2})$$

as well as relations for the triple derivatives, for example:

$$\begin{aligned} \theta_1^3 &= \frac{3(164 z_1 \theta_1 + 53568 z_1^2 \theta_1 + 20 z_1 + 1296 \theta_2 \theta_1 z_1 + 8640 z_1^2 + 3 \theta_2^2 \theta_1 + 60 \theta_2 z_1)}{\Delta_3^2}, \\ \theta_1^2 \theta_2 &= \frac{3 \theta_2 (20 z_1 + 144 z_1 \theta_1 + \theta_1 \theta_2)}{\Delta_3} \end{aligned} \quad (\text{A.3})$$

The fourth order derivatives can be expressed in terms of the Gauss-Manin connection acting on the period matrix:

$$(\theta_i - A_i(z)) \Pi(z)_\beta^\alpha = 0, \quad i = 1, \dots, h^{2,1}, \quad (\text{A.4})$$

In the following we give these matrices at the large complex structure limit for the example discussed in this work:

$$A_1(z) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{60 z_1}{\Delta_3} & \frac{432 z_1}{\Delta_3} & 0 & \frac{3}{\Delta_3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{60 z_1}{\Delta_3} & \frac{432 z_1}{\Delta_3} & 0 & \frac{3}{\Delta_3} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{3 a_1}{\Delta_3 \Delta_1} & \frac{3 a_2}{\Delta_3 \Delta_1} & \frac{3 a_3}{\Delta_3 \Delta_1} & \frac{3 a_4}{\Delta_3 \Delta_1} & \frac{60 z_1 \Delta_3^2 \Delta_2}{\Delta_1} & \frac{a_5}{\Delta_3 \Delta_1} \end{pmatrix}, \quad (\text{A.5})$$

$$A_2(z) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{a_6}{\Delta_3^2 \Delta_2} & \frac{a_7}{\Delta_3^2 \Delta_2} & \frac{a_8}{\Delta_3^2 \Delta_2} & \frac{a_9}{\Delta_3^2 \Delta_2} & \frac{-27 z_2}{\Delta_2} & \frac{a_9}{\Delta_3^2 \Delta_2} \\ \frac{a_1}{\Delta_1} & \frac{a_2}{\Delta_1} & \frac{a_3}{\Delta_1} & \frac{a_3}{\Delta_1} & \frac{60 z_1 a_9}{\Delta_1} & \frac{a_{10}}{\Delta_1} \end{pmatrix}.$$

with

$$\begin{aligned} a_1 &= 720 z_1^2 z_2 (5 + 91152 z_1), \\ a_2 &= -12 z_2 z_1 (5 - 12960 z_1 - 35645184 z_1^2), \\ a_3 &= 180 z_2 z_1 (1 - 2160 z_1 + 1679616 z_1^2), \\ a_4 &= -36 z_2 z_1 (5 - 8640 z_1 - 71103744 z_1^2), \\ a_5 &= 432 z_1 (\Delta_1(z) + 30233088 z_1^2 z_2), \\ a_6 &= -120 z_1 z_2 (1 - 864 z_1), \\ a_7 &= -2 z_2 (1 - 1266 z_1 + 546912 z_1^2), \\ a_8 &= -6 z_2 (1 - 804 z_1 + 147744 z_1^2), \\ a_9 &= 9 z_2 (1 - 1296 z_1 + 559872 z_1^2), \\ a_{10} &= 4353564672 z_1^3 z_2. \end{aligned} \quad (\text{A.6})$$

A fundamental solution is given by

$$\begin{aligned}
\pi_0(z) &= s_0(z), \\
\pi_1(z) &= s_0(z) \log z_1 + s_1(z), \\
\pi_2(z) &= s_0(z) \log z_2 + s_2(z), \\
\pi_3(z) &= s_0(z) \left(\frac{9}{2} (\log z_1)^2 + 3 \log z_1 \log z_2 \right) + s_1(z) \log z_2 + s_2(z) \log z_1 + s_3(z), \\
\pi_4(z) &= s_0(z) \left(\frac{9}{2} (\log z_1)^2 + 3 \log z_1 \log z_2 + \frac{1}{2} (\log z_1)^2 \right) \\
&\quad + s_2(z) (3 \log z_1 + \log z_2) + s_4(z), \\
\pi_5(z) &= s_0(z) \left(\frac{3}{2} (\log z_1)^3 + \frac{3}{2} (\log z_1)^2 \log z_2 + \frac{1}{2} \log z_1 (\log z_2)^2 \right) \\
&\quad + \frac{s_1(z)}{2} (\log z_2)^2 + s_2(z) \left(\frac{3}{2} (\log z_1)^2 + \log z_1 \log z_2 \right) \\
&\quad + s_3(z) \log z_2 + s_4(z) \log z_1 + s_5(z),
\end{aligned} \tag{A.7}$$

where

$$\begin{aligned}
s_0(z) &= 1 + \frac{5}{36} z_1 + \frac{385}{5184} z_1^2 + O(z^3), \\
s_1(z) &= \frac{13}{18} z_1 - \frac{2}{27} z_2 + \frac{719}{1728} z_1^2 - \frac{5}{243} z_2^2 + \frac{5}{972} z_2 z_1 + O(z^3), \\
s_2(z) &= \frac{5}{12} z_1 + \frac{2}{9} z_2 + \frac{385}{1152} z_1^2 + \frac{5}{81} z_2^2 - \frac{5}{324} z_2 z_1 + O(z^3), \\
s_3(z) &= -\frac{1}{3} z_2 + \frac{13}{4} z_1^2 - \frac{47}{324} z_2^2 + O(z^3), \\
s_4(z) &= \frac{15}{4} z_1 + \frac{10183}{768} z_1^2 + O(z^3), \\
s_5(z) &= -\frac{15}{2} z_1 + \frac{2}{3} z_2 - \frac{965}{256} z_1^2 + \frac{13}{108} z_2^2 - \frac{5}{108} z_2 z_1 + O(z^3).
\end{aligned} \tag{A.8}$$

B Holomorphic ambiguity

$$\begin{aligned}
f^{(2)}(z) = & \frac{1}{155520} \left(-111885 \bar{z}_1 + 25523 \bar{z}_2 + 671310 \bar{z}_1^2 + 111447 \bar{z}_2 \bar{z}_1 - 56842 \bar{z}_2^2 \right. \\
& - 1678275 \bar{z}_1^3 - 1204665 \bar{z}_2 \bar{z}_1^2 + 148602 \bar{z}_2^2 \bar{z}_1 + 29375 \bar{z}_2^3 + 2237700 \bar{z}_1^4 \\
& + 3455528 \bar{z}_2 \bar{z}_1^3 + 302070 \bar{z}_2^2 \bar{z}_1^2 - 136500 \bar{z}_2^3 \bar{z}_1 - 1678275 \bar{z}_1^5 - 5125329 \bar{z}_2 \bar{z}_1^4 \\
& - 1693290 \bar{z}_2^2 \bar{z}_1^3 + 202125 \bar{z}_2^3 \bar{z}_1^2 + 671310 \bar{z}_1^6 + 4481781 \bar{z}_2 \bar{z}_1^5 + 3357810 \bar{z}_2^2 \bar{z}_1^4 \\
& - 107721 \bar{z}_2^3 \bar{z}_1^3 - 111885 \bar{z}_1^7 - 2233705 \bar{z}_2 \bar{z}_1^6 - 3969738 \bar{z}_2^2 \bar{z}_1^5 - 390927 \bar{z}_2^3 \bar{z}_1^4 \\
& + 58750 \bar{z}_2^4 \bar{z}_1^3 + 489420 \bar{z}_2 \bar{z}_1^7 + 2634295 \bar{z}_2^2 \bar{z}_1^6 + 1228482 \bar{z}_2^3 \bar{z}_1^5 - 96750 \bar{z}_2^4 \bar{z}_1^4 \\
& - 836700 \bar{z}_2^2 \bar{z}_1^7 - 1223340 \bar{z}_2^3 \bar{z}_1^6 - 62250 \bar{z}_2^4 \bar{z}_1^5 + 692430 \bar{z}_2^3 \bar{z}_1^7 + 122065 \bar{z}_2^4 \bar{z}_1^6 \\
& \left. - 273015 \bar{z}_2^4 \bar{z}_1^7 + 29375 \bar{z}_2^5 \bar{z}_1^6 + 39750 \bar{z}_2^5 \bar{z}_1^7 \right) \Delta_1^{-2} \Delta_2^{-2} \quad (B.1)
\end{aligned}$$

$$\begin{aligned}
f^{(3)}(z) = & -\frac{1}{38093690880} \left(-15917050800 \bar{z}_1 + 456232932 \bar{z}_2 + 192660441750 \bar{z}_1^2 \right. \\
& + 62590386030 \bar{z}_2 \bar{z}_1 + 211279484 \bar{z}_2^2 - 1070395338600 \bar{z}_1^3 - 794525009166 \bar{z}_2 \bar{z}_1^2 \\
& - 114611573748 \bar{z}_2^2 \bar{z}_1 - 7115156792 \bar{z}_2^3 + 3611036097900 \bar{z}_1^4 \\
& + 4485991204548 \bar{z}_2 \bar{z}_1^3 + 1373729024769 \bar{z}_2^2 \bar{z}_1^2 + 172908712632 \bar{z}_2^3 \bar{z}_1 \\
& + 12595354536 \bar{z}_2^4 - 8243223219000 \bar{z}_1^5 - 15328771143252 \bar{z}_2 \bar{z}_1^4 \\
& - 7619382247178 \bar{z}_2^2 \bar{z}_1^3 - 1534203320118 \bar{z}_2^3 \bar{z}_1^2 - 182097732804 \bar{z}_2^4 \bar{z}_1 \\
& - 8683469900 \bar{z}_2^5 + 13425941147850 \bar{z}_1^6 + 35631125168634 \bar{z}_2 \bar{z}_1^5 \\
& + 25991656710522 \bar{z}_2^2 \bar{z}_1^4 + 7513251658918 \bar{z}_2^3 \bar{z}_1^3 + 1210003720515 \bar{z}_2^4 \bar{z}_1^2 \\
& + 107250300570 \bar{z}_2^5 \bar{z}_1 + 2195637500 \bar{z}_2^6 - 16018774002000 \bar{z}_1^7 \\
& - 59707988600022 \bar{z}_2 \bar{z}_1^6 - 61303837831056 \bar{z}_2^2 \bar{z}_1^5 - 24166432738356 \bar{z}_2^3 \bar{z}_1^4 \\
& - 4928943313826 \bar{z}_2^4 \bar{z}_1^3 - 611530831590 \bar{z}_2^5 \bar{z}_1^2 - 26041575000 \bar{z}_2^6 \bar{z}_1 \\
& + 14136293140200 \bar{z}_1^8 + 74311755828120 \bar{z}_2 \bar{z}_1^7 + 106181883486822 \bar{z}_2^2 \bar{z}_1^6 \\
& + 56186770195008 \bar{z}_2^3 \bar{z}_1^5 + 14124546987582 \bar{z}_2^4 \bar{z}_1^4 + 2183901301478 \bar{z}_2^5 \bar{z}_1^3 \\
& + 141417906000 \bar{z}_2^6 \bar{z}_1^2 - 9190359208800 \bar{z}_1^9 - 69493182032628 \bar{z}_2 \bar{z}_1^8 \\
& - 139262199819120 \bar{z}_2^2 \bar{z}_1^7 - 99607604872014 \bar{z}_2^3 \bar{z}_1^6 - 31105605508380 \bar{z}_2^4 \bar{z}_1^5 \\
& - 5666669637756 \bar{z}_2^5 \bar{z}_1^4 - 499739411500 \bar{z}_2^6 \bar{z}_1^3 + 4321388090250 \bar{z}_1^{10} \\
& + 48656803865922 \bar{z}_2 \bar{z}_1^9 + 139798606371588 \bar{z}_2^2 \bar{z}_1^8 + 137955097456758 \bar{z}_2^3 \bar{z}_1^7 \\
& + 55201398291783 \bar{z}_2^4 \bar{z}_1^6 + 11744735794614 \bar{z}_2^5 \bar{z}_1^5 + 1356336265200 \bar{z}_2^6 \bar{z}_1^4 \\
& + 8782550000 \bar{z}_2^7 \bar{z}_1^3 - 1414808425800 \bar{z}_1^{11} - 25014405127866 \bar{z}_2 \bar{z}_1^{10} \\
& \left. - 106841517162632 \bar{z}_2^2 \bar{z}_1^9 - 149905707199956 \bar{z}_2^3 \bar{z}_1^8 - 79956636322806 \bar{z}_2^4 \bar{z}_1^7 \right) \quad (B.2)
\end{aligned}$$

$$\begin{aligned}
& - 20332329285174 \bar{z}_2^5 \bar{z}_1^6 - 2995433412300 \bar{z}_2^6 \bar{z}_1^5 - 77818650000 \bar{z}_2^7 \bar{z}_1^4 \\
& + 300289531500 \bar{z}_1^{12} + 9067187221092 \bar{z}_2 \bar{z}_1^{11} + 60845356108857 \bar{z}_2^2 \bar{z}_1^{10} \\
& + 126488366264360 \bar{z}_2^3 \bar{z}_1^9 + 93969651592314 \bar{z}_2^4 \bar{z}_1^8 + 29701731464910 \bar{z}_2^5 \bar{z}_1^7 \\
& + 5375737341495 \bar{z}_2^6 \bar{z}_1^6 + 305868024000 \bar{z}_2^7 \bar{z}_1^5 - 35787036600 \bar{z}_1^{13} \\
& - 2148868232604 \bar{z}_2 \bar{z}_1^{12} - 24743599592694 \bar{z}_2^2 \bar{z}_1^{11} - 80849417068920 \bar{z}_2^3 \bar{z}_1^{10} \\
& - 88089834084720 \bar{z}_2^4 \bar{z}_1^9 - 36636047127000 \bar{z}_2^5 \bar{z}_1^8 - 7904357952642 \bar{z}_2^6 \bar{z}_1^7 \\
& - 752243946300 \bar{z}_2^7 \bar{z}_1^6 + 1655832150 \bar{z}_1^{14} + 286405678230 \bar{z}_2 \bar{z}_1^{13} \\
& + 6642190971806 \bar{z}_2^2 \bar{z}_1^{12} + 37427283757680 \bar{z}_2^3 \bar{z}_1^{11} + 63913185937407 \bar{z}_2^4 \bar{z}_1^{10} \\
& + 37575505804186 \bar{z}_2^5 \bar{z}_1^9 + 9831162295782 \bar{z}_2^6 \bar{z}_1^8 + 1360789452540 \bar{z}_2^7 \bar{z}_1^7 \\
& + 13173825000 \bar{z}_2^8 \bar{z}_1^6 - 14575439970 \bar{z}_2 \bar{z}_1^{14} - 1005306836100 \bar{z}_2^2 \bar{z}_1^{13} \\
& - 11572589903500 \bar{z}_2^3 \bar{z}_1^{12} - 34202435930730 \bar{z}_2^4 \bar{z}_1^{11} - 30897046296546 \bar{z}_2^5 \bar{z}_1^{10} \\
& - 10439016904684 \bar{z}_2^6 \bar{z}_1^9 - 1915988002740 \bar{z}_2^7 \bar{z}_1^8 - 77206500000 \bar{z}_2^8 \bar{z}_1^7 \\
& + 56821108680 \bar{z}_2^2 \bar{z}_1^{14} + 2028431619060 \bar{z}_2^3 \bar{z}_1^{13} + 12418135213655 \bar{z}_2^4 \bar{z}_1^{12} \\
& + 19333958170350 \bar{z}_2^5 \bar{z}_1^{11} + 9305421434772 \bar{z}_2^6 \bar{z}_1^{10} + 2108401264068 \bar{z}_2^7 \bar{z}_1^9 \\
& + 187661061000 \bar{z}_2^8 \bar{z}_1^8 - 129039404760 \bar{z}_2^3 \bar{z}_1^{14} - 2587173466500 \bar{z}_2^4 \bar{z}_1^{13} \\
& - 8403448711600 \bar{z}_2^5 \bar{z}_1^{12} - 6725788007592 \bar{z}_2^6 \bar{z}_1^{11} - 1892891215014 \bar{z}_2^7 \bar{z}_1^{10} \\
& - 277132387700 \bar{z}_2^8 \bar{z}_1^9 + 188761664700 \bar{z}_2^4 \bar{z}_1^{14} + 2155595370600 \bar{z}_2^5 \bar{z}_1^{13} \\
& + 3522052783964 \bar{z}_2^6 \bar{z}_1^{12} + 1456170527574 \bar{z}_2^7 \bar{z}_1^{11} + 293531487060 \bar{z}_2^8 \bar{z}_1^{10} \\
& + 8782550000 \bar{z}_2^9 \bar{z}_1^9 - 185488839900 \bar{z}_2^5 \bar{z}_1^{14} - 1165840766580 \bar{z}_2^6 \bar{z}_1^{13} \\
& - 868288332856 \bar{z}_2^7 \bar{z}_1^{12} - 221357393880 \bar{z}_2^8 \bar{z}_1^{11} - 25123350000 \bar{z}_2^9 \bar{z}_1^{10} \\
& + 123701472720 \bar{z}_2^6 \bar{z}_1^{14} + 389322265500 \bar{z}_2^7 \bar{z}_1^{13} + 124275425135 \bar{z}_2^8 \bar{z}_1^{12} \\
& + 23389674000 \bar{z}_2^9 \bar{z}_1^{11} - 55116605880 \bar{z}_2^7 \bar{z}_1^{14} - 69934264260 \bar{z}_2^8 \bar{z}_1^{13} \\
& - 15944383000 \bar{z}_2^9 \bar{z}_1^{12} + 15644258910 \bar{z}_2^8 \bar{z}_1^{14} + 3981361650 \bar{z}_2^9 \bar{z}_1^{13} \\
& + 2195637500 \bar{z}_2^{10} \bar{z}_1^{12} - 2542777650 \bar{z}_2^9 \bar{z}_1^{14} + 306075000 \bar{z}_2^{10} \bar{z}_1^{13} \\
& + 178731000 \bar{z}_2^{10} \bar{z}_1^{14}) \Delta_1^{-4} \Delta_2^{-4}
\end{aligned}$$

C GV invariants

$d_1 \setminus d_2$	0	1	2	3	4	5	6	7
0	0	540	540	540	540	540	540	540
1	3	-1080	143370	204071184	21772947555	1076518252152	33381348217290	746807207168880
2	-6	2700	-574560	74810520	-49933059660	7772494870800	31128163315047072	8211715737128556480
3	27	-17280	5051970	-913383000	224108858700	-42712135606368	4047949393968960	-166123333123572659520
4	-192	154440	-57879900	13593850920	-2953943334360	603778002921828	-90433961251273800	50057390316302661600
5	1695	-1640520	751684050	-218032516800	51350781706785	-11035406089270080	2000248139674298880	-541531457497667187360
6	-17064	19369800	-10500261120	3630383423100	-967920854160960	224651517028866252	-45689218327425589920	10071417619296745378920
7	188454	-245635200	153827405370	-61789428573120	18707398902511245	-4765797079033190400	1064787655240073455400	-230224103349955979141880

Table 3: $g = 0$

$d_1 \setminus d_2$	0	1	2	3	4	5	6	7
0	0	3	0	0	0	0	0	0
1	0	-6	2142	-280284	-408993990	-44771454090	-2285308753398	-73398848219076
2	0	15	-8568	2126358	521854854	1122213103092	879831736792200	205929022209626928
3	-10	4764	-1079298	152278992	-16704086880	-3328467399468	1252978673849946	-556349234873466744
4	231	-154662	48907800	-9759419622	1591062429648	-186415241060547	8624795296820760	2067149471538742920
5	-4452	3762246	-1510850250	385304916960	-76672173887766	12768215950604070	-1663415916220743876	220904813068869853736
6	80958	-82308270	40028268276	-12433493287620	2931354541290318	-578520552756118977	96321811855350031992	-15333848730658632865302
7	-1438086	1707634920	-974938365558	357248310744312	-97937943585729324	2214402244440264176	-4288880126137360757400	762495977216972967628344

Table 4: $g = 1$

$d_1 \setminus d_2$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	9	-3192	412965	614459160	68590330119	3587118690600
2	0	0	-36	20826	-5904756	-47646003780	-80065270602672	-36393689644146360
3	0	27	-16884	4768830	-818096436	288137120463	67873415627151	45583988161896702
4	-102	57456	-15452514	2632083714	-320511624876	18550698291252	780000198300540	-251496603078253344
5	5430	-4032288	1430896428	-323858122812	55058565096630	-7249216518163620	691264676523200805	-39745849558901142924
6	-194022	177495894	-7787279952	21874076033328	-4595039844606324	780316191323388252	-108001731472892477172	12700932052931799955182
7	5784837	-6277761198	3280241914893	-1101478942766574	274831572910592142	-55535640852991791852	9409679296993051279011	-1395184265801287057499886

Table 5: $g = 2$

$d_1 \setminus d_2$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	-12	4230	-541440	-820457286	-93230754660
2	0	0	0	66	-45729	627574428	3776946955338	3154704156093636
3	0	0	-72	48036	-14756490	297044064	-7900517344212	-6362380878728364
4	15	-7236	1638918	-226431351	20419274259	-719284158099	236091664016826	-65579326297771734
5	-3672	2417742	-764921214	154856849136	-22866882491772	2493418732350750	-194361733345447458	11145513580945101792
6	290853	-240662448	95825798874	-24497597694651	4625681034438657	-687273507534149145	81177932356719743208	-7626799243005256969200
7	-15363990	15291362682	-7342199188434	2269518622807320	-518201778138767424	94360213143715120002	-14152475137808110529514	1791485123982485159044584

Table 6: $g = 3$

D Modular forms

D.1 Definitions

We summarize the definitions of the modular objects appearing in this work.

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad \Delta(\tau) = \eta(\tau)^{24} \quad (\text{D.1})$$

and transforms according to

$$\eta(\tau + 1) = e^{\frac{i\pi}{12}} \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}} \eta(\tau). \quad (\text{D.2})$$

The Eisenstein series are defined by

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n}, \quad (\text{D.3})$$

where B_k denotes the k -th Bernoulli number. E_k is a modular form of weight k for $k > 2$ and even. The discriminant form is

$$\Delta(\tau) = \frac{1}{1728} (E_4(\tau)^3 - E_6(\tau)^2) = \eta(\tau)^{24}. \quad (\text{D.4})$$

The modular completion of the holomorphic Eisenstein series E_2 has the form

$$\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi \text{Im}\tau}. \quad (\text{D.5})$$

D.2 Expansions of $f_n^{(g)}$

$$f_1^{(0)} = \frac{1}{48} \Delta^{-\frac{3}{2}} E_4 (113 E_6^2 + 31 E_4^3) \quad (\text{D.6})$$

$$f_2^{(0)} = \frac{1}{221184} \Delta^{-3} (E_4 E_6 (196319 E_6^4 + 755906 E_6^2 E_4^3 + 208991 E_4^6) + 4 E_4^2 (113 E_6^2 + 31 E_4^3)^2 E_2) \quad (\text{D.7})$$

$$f_3^{(0)} = \frac{1}{557256278016} \Delta^{-\frac{9}{2}} (E_4 (360744024241 E_6^8 + 4311836724416 E_6^6 E_4^3 + 6966210848730 E_6^4 E_4^6 + 1904214859592 E_6^2 E_4^9 + 49789907821 E_4^{12}) + 8748 E_4^2 E_6 (113 E_6^2 + 31 E_4^3) (196319 E_6^4 + 755906 E_6^2 E_4^3 + 208991 E_4^6) E_2 + 17496 E_4^3 (113 E_6^2 + 31 E_4^3)^3 E_2^2) \quad (\text{D.8})$$

$$f_1^{(1)} = \frac{1}{576} \Delta^{-\frac{3}{2}} E_4 (113 E_6^2 + 31 E_4^3) E_2 \quad (\text{D.9})$$

$$f_2^{(1)} = \frac{1}{31850496} \Delta^{-3} \left(-1322175 E_6^4 E_4^3 - 1941621 E_6^2 E_4^6 - 21935 E_6^6 - 197917 E_4^9 \right. \\ \left. + 12 E_4 E_6 (196319 E_6^4 + 755906 E_6^2 E_4^3 + 208991 E_4^6) E_2 \right. \\ \left. + 72 E_4^2 (113 E_6^2 + 31 E_4^3)^2 E_2^2 \right) \quad (D.10)$$

$$f_3^{(1)} = \frac{1}{743008370688} \Delta^{-\frac{9}{2}} \left(8 E_6 (87737816690 E_6^6 E_4^3 + 355811791488 E_6^4 E_4^6 \right. \\ \left. + 255154185422 E_6^2 E_4^9 + 28404078217 E_4^{12} + 1388616631 E_6^8) \right. \\ \left. + 81 E_4 (113 E_6^2 + 31 E_4^3) (1322175 E_6^4 E_4^3 + 1941621 E_6^2 E_4^6 \right. \\ \left. + 21935 E_6^6 + 197917 E_4^9) E_2 - 972 E_4^2 E_6 (113 E_6^2 + 31 E_4^3) \right. \\ \left. (196319 E_6^4 + 755906 E_6^2 E_4^3 + 208991 E_4^6) E_2^2 \right. \\ \left. - 3240 E_4^3 (113 E_6^2 + 31 E_4^3)^3 E_2^3 \right) \quad (D.11)$$

$$f_1^{(2)} = \frac{1}{69120} \Delta^{-\frac{3}{2}} \left(E_4^2 (113 E_6^2 + 31 E_4^3) + 5 E_4 (113 E_6^2 + 31 E_4^3) E_2^2 \right) \quad (D.12)$$

$$f_2^{(2)} = \frac{1}{1911029760} \Delta^{-3} \left(2 E_4^2 E_6 (1540871 E_6^4 + 7232114 E_6^2 E_4^3 + 3336839 E_4^6) \right. \\ \left. + (9371817 E_6^2 E_4^6 + 5997963 E_4^3 E_6^4 + 943457 E_4^9 + 109675 E_6^6) E_2 \right. \\ \left. - 30 E_4 E_6 (196319 E_6^4 + 755906 E_6^2 E_4^3 + 208991 E_4^6) E_2^2 \right. \\ \left. - 280 E_4^2 (113 E_6^2 + 31 E_4^3)^2 E_2^3 \right) \quad (D.13)$$

$$f_3^{(2)} = \frac{1}{4953389137920} \Delta^{-\frac{9}{2}} \left(2 E_4^2 (6841970275 E_6^8 + 59257855181 E_6^2 E_4^9 \right. \\ \left. + 188946594537 E_6^4 E_4^6 + 103842683975 E_6^6 E_4^3 + 1946160544 E_4^{12}) \right. \\ \left. - 36 E_4^3 E_6 (113 E_6^2 + 31 E_4^3) (1354933 E_4^6 + 2482198 E_6^2 E_4^3 + 475957 E_6^4) E_2 \right. \\ \left. - 9 E_4 (113 E_6^2 + 31 E_4^3) (954989 E_4^9 + 9455889 E_6^2 E_4^6 + 6151191 E_4^3 E_6^4 \right. \\ \left. + 109675 E_6^6) E_2^2 + 360 E_4^2 E_6 (113 E_6^2 + 31 E_4^3) (196319 E_6^4 + 755906 E_6^2 E_4^3 \right. \\ \left. + 208991 E_4^6) E_2^3 + 1710 E_4^3 (113 E_6^2 + 31 E_4^3)^3 E_2^4 \right) \quad (D.14)$$

$$f_1^{(3)} = \frac{1}{17418240} \Delta^{-\frac{3}{2}} \left(4 E_4 E_6 (113 E_6^2 + 31 E_4^3) + 21 E_4^2 (113 E_6^2 + 31 E_4^3) E_2 \right. \\ \left. + 35 E_4 (113 E_6^2 + 31 E_4^3) E_2^3 \right) \quad (D.15)$$

$$f_2^{(3)} = \frac{1}{321052999680} \Delta^{-3} \left(E_4 (14470511 E_6^6 + 299836579 E_4^3 E_6^4 + 378756589 E_6^2 E_4^6 \right. \\ \left. + 31120385 E_4^9) + 12 E_4^2 E_6 (3459163 E_6^4 + 16800202 E_6^2 E_4^3 + 7775707 E_4^6) E_2 \right. \\ \left. + (767725 E_6^6 + 5958407 E_4^9 + 33404973 E_4^3 E_6^4 + 60894687 E_6^2 E_4^6) E_2^2 \right. \\ \left. - 140 E_4 E_6 (196319 E_6^4 + 755906 E_6^2 E_4^3 + 208991 E_4^6) E_2^3 \right. \\ \left. - 2100 E_4^2 (113 E_6^2 + 31 E_4^3)^2 E_2^4 \right) \quad (D.16)$$

$$f_3^{(3)} = \frac{1}{624127031377920} \Delta^{-\frac{9}{2}} \left(2 E_4 E_6 (42089002745 E_6^8 + 856373539390 E_6^6 E_4^3 \right.$$

$$\begin{aligned}
& +2773682486544 E_6^4 E_4^6 + 2005074999106 E_6^2 E_4^9 + 260719698551 E_4^{12}) \\
& - 27 E_4^2 (113 E_6^2 + 31 E_4^3) (8126451 E_4^9 + 97251020 E_6^2 E_4^6 \\
& + 74249327 E_4^3 E_6^4 + 2870738 E_6^6) E_2 - 54 E_4^3 E_6 (113 E_6^2 + 31 E_4^3) (9472999 E_4^6 \\
& + 17291314 E_6^2 E_4^3 + 3178471 E_6^4) E_2^2 - 315 E_4 (113 E_6^2 + 31 E_4^3) (180619 E_4^9 \\
& + 1815513 E_6^2 E_4^6 + 1092333 E_4^3 E_6^4 + 21935 E_6^6) E_2^3 \\
& + 1890 E_4^2 E_6 (113 E_6^2 + 31 E_4^3) (196319 E_6^4 + 755906 E_6^2 E_4^3 + 208991 E_4^6) E_2^4 \\
& + 12285 E_4^3 (113 E_6^2 + 31 E_4^3)^3 E_2^5) \tag{D.17}
\end{aligned}$$

Bibliography

- [1] W. Lerche, C. Vafa, and N. P. Warner, “Chiral Rings in N=2 Superconformal Theories,” *Nucl.Phys.* **B324** (1989) 427.
- [2] R. Gopakumar and C. Vafa, “M theory and topological strings. 1.,” [arXiv:hep-th/9809187](#) [hep-th].
- [3] R. Gopakumar and C. Vafa, “M theory and topological strings. 2.,” [arXiv:hep-th/9812127](#) [hep-th].
- [4] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, “Holomorphic anomalies in topological field theories,” *Nucl.Phys.* **B405** (1993) 279–304, [arXiv:hep-th/9302103](#) [hep-th].
- [5] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, “Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes,” *Commun.Math.Phys.* **165** (1994) 311–428, [arXiv:hep-th/9309140](#) [hep-th].
- [6] E. Witten, “Quantum background independence in string theory,” [arXiv:hep-th/9306122](#) [hep-th].
- [7] I. Antoniadis and S. Hohenegger, “Topological amplitudes and physical couplings in string theory,” *Nucl.Phys.Proc.Suppl.* **171** (2007) 176–195, [arXiv:hep-th/0701290](#) [HEP-TH].
- [8] H. Ooguri, A. Strominger, and C. Vafa, “Black hole attractors and the topological string,” *Phys.Rev.* **D70** (2004) 106007, [arXiv:hep-th/0405146](#) [hep-th].
- [9] R. E. Rudd, “The String partition function for QCD on the torus,” [arXiv:hep-th/9407176](#) [hep-th].
- [10] R. Dijkgraaf, “Mirror symmetry and elliptic curves,” in *The moduli space of curves (Texel Island, 1994)*, vol. 129 of *Progr. Math.*, pp. 149–163. Birkhäuser Boston, Boston, MA, 1995.

- [11] M. Kaneko and D. Zagier, “A generalized Jacobi theta function and quasimodular forms,” in *The moduli space of curves (Texel Island, 1994)*, vol. 129 of *Progr. Math.*, pp. 165–172. Birkhäuser Boston, Boston, MA, 1995.
- [12] J. Minahan, D. Nemeschansky, C. Vafa, and N. Warner, “E strings and N=4 topological Yang-Mills theories,” *Nucl.Phys.* **B527** (1998) 581–623, [arXiv:hep-th/9802168](#) [[hep-th](#)].
- [13] S. Hosono, M. Saito, and A. Takahashi, “Holomorphic anomaly equation and BPS state counting of rational elliptic surface,” *Adv.Theor.Math.Phys.* **3** (1999) 177–208, [arXiv:hep-th/9901151](#) [[hep-th](#)].
- [14] S. Hosono, “Counting BPS states via holomorphic anomaly equations,” *Fields Inst.Comm.* (2002) 57–86, [arXiv:hep-th/0206206](#) [[hep-th](#)].
- [15] A. Klemm, M. Kreuzer, E. Riegler, and E. Scheidegger, “Topological string amplitudes, complete intersection Calabi-Yau spaces and threshold corrections,” *JHEP* **0505** (2005) 023, [arXiv:hep-th/0410018](#) [[hep-th](#)].
- [16] M. Aganagic, V. Bouchard, and A. Klemm, “Topological Strings and (Almost) Modular Forms,” *Commun.Math.Phys.* **277** (2008) 771–819, [arXiv:hep-th/0607100](#) [[hep-th](#)].
- [17] M. Gunaydin, A. Neitzke, and B. Pioline, “Topological wave functions and heat equations,” *JHEP* **0612** (2006) 070, [arXiv:hep-th/0607200](#) [[hep-th](#)].
- [18] T. W. Grimm, A. Klemm, M. Marino, and M. Weiss, “Direct Integration of the Topological String,” *JHEP* **0708** (2007) 058, [arXiv:hep-th/0702187](#) [[HEP-TH](#)].
- [19] S. Yamaguchi and S.-T. Yau, “Topological string partition functions as polynomials,” *JHEP* **0407** (2004) 047, [arXiv:hep-th/0406078](#) [[hep-th](#)].
- [20] M. Alim and J. D. Lange, “Polynomial Structure of the (Open) Topological String Partition Function,” *JHEP* **0710** (2007) 045, [arXiv:0708.2886](#) [[hep-th](#)].
- [21] J. Minahan, D. Nemeschansky, and N. Warner, “Partition functions for BPS states of the noncritical E(8) string,” *Adv.Theor.Math.Phys.* **1** (1998) 167–183, [arXiv:hep-th/9707149](#) [[hep-th](#)].
- [22] J. Minahan, D. Nemeschansky, and N. Warner, “Instanton expansions for mass deformed N=4 superYang-Mills theories,” *Nucl.Phys.* **B528** (1998) 109–132, [arXiv:hep-th/9710146](#) [[hep-th](#)].
- [23] P. Candelas, X. C. De La Ossa, P. S. Green, and L. Parkes, “A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory,” *Nucl.Phys.* **B359** (1991) 21–74.
- [24] V. V. Batyrev, “Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties,” [arXiv:alg-geom/9310003](#) [[alg-geom](#)].

- [25] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau, “Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces,” *Commun.Math.Phys.* **167** (1995) 301–350, [arXiv:hep-th/9308122](#) [hep-th].
- [26] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, vol. 68 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.
- [27] M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens, and M. Soroush, “Hints for Off-Shell Mirror Symmetry in type II/F-theory Compactifications,” *Nucl.Phys.* **B841** (2010) 303–338, [arXiv:0909.1842](#) [hep-th].
- [28] M. Alim, M. Hecht, H. Jockers, P. Mayr, A. Mertens, and M. Soroush, “Flat Connections in Open String Mirror Symmetry,” [arXiv:1110.6522](#) [hep-th].
- [29] W. Lerche, D. Smit, and N. Warner, “Differential equations for periods and flat coordinates in two-dimensional topological matter theories,” *Nucl.Phys.* **B372** (1992) 87–112, [arXiv:hep-th/9108013](#) [hep-th].
- [30] S. Ferrara and J. Louis, “Flat holomorphic connections and Picard-Fuchs identities from N=2 supergravity,” *Phys.Lett.* **B278** (1992) 240–245, [arXiv:hep-th/9112049](#) [hep-th].
- [31] A. Ceresole, R. D’Auria, S. Ferrara, W. Lerche, and J. Louis, “Picard-Fuchs equations and special geometry,” *Int.J.Mod.Phys.* **A8** (1993) 79–114, [arXiv:hep-th/9204035](#) [hep-th].
- [32] A. Ceresole, R. D’Auria, S. Ferrara, W. Lerche, J. Louis, and T. Regge, “Picard-Fuchs equations, special geometry and target space duality,”. Contribution to second volume of ‘Essays on Mirror Manifolds’.
- [33] S. Hosono, B. Lian, and S.-T. Yau, “GKZ generalized hypergeometric systems in mirror symmetry of Calabi-Yau hypersurfaces,” *Commun.Math.Phys.* **182** (1996) 535–578, [arXiv:alg-geom/9511001](#) [alg-geom].
- [34] S. Hosono and B. Lian, “GKZ hypergeometric systems and applications to mirror symmetry,” [arXiv:hep-th/9602147](#) [hep-th].
- [35] M. Noguchi, “Mirror symmetry of Calabi-Yau manifolds and flat coordinates,” *Int.J.Mod.Phys.* **A12** (1997) 4973–4996, [arXiv:hep-th/9609163](#) [hep-th].
- [36] T. Masuda and H. Suzuki, “Prepotentials, bilinear forms on periods and enhanced gauge symmetries in Type II strings,” *Int.J.Mod.Phys.* **A14** (1999) 1177–1204, [arXiv:hep-th/9807062](#) [hep-th].
- [37] E. Witten, “Phases of N=2 theories in two-dimensions,” *Nucl.Phys.* **B403** (1993) 159–222, [arXiv:hep-th/9301042](#) [hep-th].
- [38] P. Candelas, A. Font, S. H. Katz, and D. R. Morrison, “Mirror symmetry for two parameter models. 2.,” *Nucl.Phys.* **B429** (1994) 626–674, [arXiv:hep-th/9403187](#) [hep-th].

- [39] K. Hori and C. Vafa, “Mirror symmetry,” [arXiv:hep-th/0002222](#) [hep-th].
- [40] V. V. Batyrev and D. van Straten, “Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties,” *Commun.Math.Phys.* **168** (1995) 493–534, [arXiv:alg-geom/9307010](#) [alg-geom].
- [41] I. Gelfand, M. Kapranov, and A. Zelevinsky, “Hypergeometric functions and toric varieties,” *Funct. Anal. Appl.* **23** (1989) .
- [42] P. Deligne, *Équations différentielles à points singuliers réguliers*. Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin, 1970.
- [43] P. Candelas and X. de la Ossa, “Moduli Space of Calabi-Yau Manifolds,” *Nucl.Phys.* **B355** (1991) 455–481.
- [44] A. Strominger, “Special Geometry,” *Commun.Math.Phys.* **133** (1990) 163–180.
- [45] A. Klemm, B. Lian, S. Roan, and S.-T. Yau, “A Note on ODEs from mirror symmetry,” [arXiv:hep-th/9407192](#) [hep-th].
- [46] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau, “Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces,” *Nucl.Phys.* **B433** (1995) 501–554, [arXiv:hep-th/9406055](#) [hep-th].
- [47] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.
- [48] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers, *Algebraic D-modules*, vol. 2 of *Perspectives in Mathematics*. Academic Press Inc., Boston, MA, 1987.
- [49] M. Alim, J. D. Lange, and P. Mayr, “Global Properties of Topological String Amplitudes and Orbifold Invariants,” *JHEP* **1003** (2010) 113, [arXiv:0809.4253](#) [hep-th].
- [50] S. Hosono, “BCOV ring and holomorphic anomaly equation,” [arXiv:0810.4795](#) [math.AG].
- [51] A. Landman, “On the Picard-Lefschetz transformation for algebraic manifolds acquiring general singularities,” *Trans. Amer. Math. Soc.* **181** (1973) 89–126.
- [52] M. Marino and G. W. Moore, “Counting higher genus curves in a Calabi-Yau manifold,” *Nucl.Phys.* **B543** (1999) 592–614, [arXiv:hep-th/9808131](#) [hep-th].
- [53] C. Faber and R. Pandharipande, “Hodge integrals and Gromov-Witten theory,” *Inventiones Mathematicae* **139** (1999) 173–199, [arXiv:98101173](#) [math.AG].
- [54] A. Klemm and P. Mayr, “Strong coupling singularities and nonAbelian gauge symmetries in N=2 string theory,” *Nucl.Phys.* **B469** (1996) 37–50, [arXiv:hep-th/9601014](#) [hep-th].

- [55] D. Ghoshal and C. Vafa, “C = 1 string as the topological theory of the conifold,” *Nucl.Phys.* **B453** (1995) 121–128, [arXiv:hep-th/9506122](#) [hep-th].
- [56] I. Antoniadis, E. Gava, K. Narain, and T. Taylor, “N=2 type II heterotic duality and higher derivative F terms,” *Nucl.Phys.* **B455** (1995) 109–130, [arXiv:hep-th/9507115](#) [hep-th].
- [57] C. Vafa, “A Stringy test of the fate of the conifold,” *Nucl.Phys.* **B447** (1995) 252–260, [arXiv:hep-th/9505023](#) [hep-th].
- [58] M. Aganagic, A. Klemm, M. Marino, and C. Vafa, “Matrix model as a mirror of Chern-Simons theory,” *JHEP* **0402** (2004) 010, [arXiv:hep-th/0211098](#) [hep-th].
- [59] M.-x. Huang and A. Klemm, “Holomorphic Anomaly in Gauge Theories and Matrix Models,” *JHEP* **0709** (2007) 054, [arXiv:hep-th/0605195](#) [hep-th].
- [60] M.-x. Huang, A. Klemm, and S. Quackenbush, “Topological string theory on compact Calabi-Yau: Modularity and boundary conditions,” *Lect.Notes Phys.* **757** (2009) 45–102, [arXiv:hep-th/0612125](#) [hep-th].
- [61] B. Haghighat and A. Klemm, “Solving the Topological String on K3 Fibrations,” *JHEP* **1001** (2010) 009, [arXiv:0908.0336](#) [hep-th]. With an appendix by Sheldon Katz.
- [62] P. Berglund and S. H. Katz, “Mirror symmetry for hypersurfaces in weighted projective space and topological couplings,” *Nucl.Phys.* **B420** (1994) 289–314, [arXiv:hep-th/9311014](#) [hep-th].
- [63] A. Klemm, P. Mayr, and C. Vafa, “BPS states of exceptional noncritical strings,” [arXiv:hep-th/9607139](#) [hep-th]. 29 pages, 1 figure Report-no: HUTP-96/A031, CERN-TH-96-184.
- [64] K. Sakai, “Topological string amplitudes for the local half K3 surface,” [arXiv:1111.3967](#) [hep-th].
- [65] C. Vafa and E. Witten, “A Strong coupling test of S duality,” *Nucl.Phys.* **B431** (1994) 3–77, [arXiv:hep-th/9408074](#) [hep-th].
- [66] D. Zagier, “Ramanujan’s Mock Theta Functions and their Applications d’après Zwegers and Bringmann-Ono,” *Séminaire BOURBAKI* **986** (2007) .
- [67] J. Manschot, “Wall-crossing of D4-branes using flow trees,” *Adv.Theor.Math.Phys.* **15** (2011) 1–42, [arXiv:1003.1570](#) [hep-th]. 41 pages, 3 figures.
- [68] J. Manschot, “The Betti numbers of the moduli space of stable sheaves of rank 3 on P^2 ,” *Lett.Math.Phys.* **98** (2011) 65–78, [arXiv:1009.1775](#) [math-ph].
- [69] M. Alim, B. Haghighat, M. Hecht, A. Klemm, M. Rauch, and T. Wotschke, “Wall-crossing holomorphic anomaly and mock modularity of multiple M5-branes,” [arXiv:1012.1608](#) [hep-th]. 45 p.

- [70] J. Manschot, “BPS invariants of semi-stable sheaves on rational surfaces,”
arXiv:1109.4861 [math-ph]. 23 pages.